UNIVERSITY OF HELSINKI REPORT SERIES IN PHYSICS

HU-P-D216

Applications of Curved Space Field Theory to Simple Scalar Field Models of Inflation

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ACADEMIC DISSERTATION

To be presented, with the permission of the Faculty of Science of the University of Helsinki, for public criticism in the auditorium E204 at Physicum, Gustaf Hällströmin katu 2A, Helsinki, on the 5th of May 2014 at 12 o'clock.

Helsinki 2014

ISBN 978-952-10-8960-2 (printed version)
ISSN 0356-0961
ISBN 978-952-10-8961-9 (pdf version)
http://ethesis.helsinki.fi
Unigrafia
Helsinki 2014

T. Markkanen: Applications of Curved Space Field Theory to Simple Scalar Field Models of Inflation, University of Helsinki, 2014, 72 pages,
University of Helsinki Report Series in Physics, HU-P-D216
ISSN 0356-0961
ISBN 978-952-10-8960-2 (printed version)
ISBN 978-952-10-8961-9 (pdf version)

Abstract

Cosmic inflation is a phase of accelerating, nearly exponential expansion of the spacetime fabric of the Universe, which is assumed to have taken place almost immediately after the Big Bang. Inflation possesses the appealing property that it provides solutions to deep cosmological problems, such as the flatness and horizon problems, and also gives a natural origin for the formation of the large scale structures we observe today.

In this thesis we set out to investigate the role quantum corrections play for some simple models where inflation is driven by a single scalar field. It is essential that here the quantum corrections are calculated via curved space field theory. In this technique one quantizes only the matter fields, the dynamics of which take place on a curved classical background. This approach is rarely used in mainstream cosmology and it has the benefit that it allows the quantum fluctuations to back-react on classical Einsteinian gravity.

The curved space quantum corrections are studied first in the effective action formalism via the Schwinger-DeWitt expansion and then by constructing effective equations of motion by using the slow-roll technique. We also focus on consistent renormalization and show how to renormalize the effective equations of motion without any reference to an effective action for an interacting theory in curved spacetime. Due to a potential infrared enhancement in effective equations in quasi-de Sitter space, we also perform a resummation of Feynman diagrams in curved non-static space and observe that it regulates the infrared effects.

Concerning implications for actual inflationary models, we focus on chaotic type models and observe the quantum corrections to be insignificant, but nevertheless to have theoretically a non-trivial structure.

Acknowledgments

This thesis is based on the research carried out at the Division of Elementary Particle Physics at the Department of Physics of the University of Helsinki. The work was in large part funded by the Finnish Academy of Science and Letters. The Niels Bohr Institute of the University of Copenhagen is acknowledged for providing support for three lengthy visits.

First and foremost I would like to thank both of my supervisors, Kari Rummukainen and Anders Tranberg, for introducing me to this interesting topic and more importantly for their insights and guidance throughout the process, especially during the early stages when help was much-needed. I am also thankful to my recent collaborator Matti Herranen for illuminating discussions and calculational assistance.

I would also like to thank my colleagues/teachers Mark Hindmarsh, Keijo Kajantie, Esko Keski-Vakkuri, Sami Nurmi, Syksy Räsänen and David Weir for always having time for discussions, physics related or otherwise. I am especially grateful to David Weir for proofreading the finished manuscript. I also wish to thank all my other co-workers at the Division of Elementary Particle Physics and the Helsinki Institute of Physics for providing a laid-back atmosphere.

I wish to thank the pre-examiners of this thesis Kimmo Kainulainen and Daniel Litim for carefully reading the manuscript and suggesting useful improvements.

I express my gratitude to Julien Serreau for agreeing to act as the opponent for my thesis defence despite his busy schedule.

Last but certainly not least I wish to deeply thank my family, friends and my significant other Jenni for support and encouragement during all those not-so-smooth stages of this project. Without you, there would have been no thesis.

Helsinki, April 2014

Tommi Markkanen

Publications

The research publications included in this thesis are:

I T. Markkanen and A. Tranberg,
Quantum Corrections to Inflaton and Curvaton Dynamics,
JCAP **1211** (2012) 027
[arXiv:1207.2179 [gr-qc]].

II T. Markkanen and A. Tranberg,
A Simple Method for One-Loop Renormalization in Curved Space-Time,
JCAP **1308** (2013) 045
[arXiv:1303.0180 [hep-th]].

III M. Herranen, T. Markkanen and A. Tranberg,
Quantum Corrections to Scalar Field Dynamics in a Slow-roll Space-time,
[arXiv:1311.5532 [hep-ph]]
Accepted for publication in JHEP in April of 2014.

Author's contribution to the joint publications

For article **I**, the author suggested the use of the Schwinger-DeWitt expansion and performed the analytical calculations. The article was then jointly written with Anders Tranberg, who also wrote the code for the numerical solution of the equations.

For article **II**, formulating the initial research problem, performing the analytical calculations as well as writing the first draft were done by the author. The draft was then polished with Anders Tranberg who also provided guidance throughout the entire process.

The research problem for article **III** came jointly from all three authors and the calculations were performed by Tommi Markkanen and Matti Herranen and the final version was jointly written by the three authors.

Units and conventions

Throughout this thesis we will use natural units, where the speed of light, the Planck constant and the Boltzmann constant are set to unity, i.e

$$c \equiv \hbar \equiv k_{\mathrm{B}} \equiv 1.$$

Furthermore we will frequently make use of the *reduced Planck mass* defined as

$$M_{\mathrm{pl}}^2 \equiv \frac{\hbar c}{8\pi G} \equiv \frac{1}{8\pi G},$$

with G being Newton's constant. Our signs are chosen according to the $(+,+,+)$ convention in the classification of [1]. This means that the Minkowski metric, the Riemann tensor and the Einstein field equation are defined respectively as

$$\begin{aligned}
\eta_{\mu\nu} &= \mathrm{diag}(-1,+1,+1,+1) \\
R^{\delta}{}_{\alpha\beta\gamma} &= \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\alpha\beta,\gamma} + \Gamma^{\delta}_{\sigma\beta}\Gamma^{\sigma}_{\gamma\alpha} - \Gamma^{\delta}_{\sigma\gamma}\Gamma^{\sigma}_{\beta\alpha} \\
G_{\mu\nu} &= \frac{1}{M_{\mathrm{pl}}^2} T_{\mu\nu}.
\end{aligned}$$

The spatial parts of vectors are denoted with boldface letters, $\mathbf{x}$ and $\mathbf{k}$ in position and momentum space respectively and for the length of the position space components we simply write $|\mathbf{k}| \equiv k$. Derivatives with respect to time are denoted with dots,

$$\frac{d}{dt} f(t) \equiv \dot{f}(t).$$

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Chapter 1

Introduction

The current understanding of the evolution of the early universe includes a phase of almost exponential expansion, named inflation. The inflationary past of our universe is not invisible to current observers since some aspects of inflation leave their imprint in the cosmic microwave background and thus can be probed with today's observations. Experiments seem to be providing evidence in support of early universe inflation. Because of this it is important to understand all its predictions, including the ones that may be currently considered unobservable, if only for the sake of theoretical consistency. Among other things, this motivates the inclusion of quantum effects for inflationary dynamics. However, quantum corrections in an inflationary setting are not a question of purely theoretical interest. Now, in the wake of the Planck results [2] and in the run-up to the Euclid mission [3], one may argue that cosmological research has entered an era where deriving high-precision results is becoming increasingly relevant also due to the high accuracy of the available experimental data. This is especially true in the context of inflation, where distinguishing the correct model is still at the moment an open issue. Currently, there are a number of models that are in accord with the most up to date results [4], a fact that may be challenged by the precision of proposed future missions [5, 6]. Until recently it has not been very typical to perform cosmological calculations in a consistent field theory setting, as quite often the classical results are assumed to suffice. From a quantitative perspective this is understandable, since often such efforts are of little importance for measurable results. It is also often the case that the amount of work required in deriving the fully quantum field theoretic result surpasses greatly that needed for the classical derivation. This is especially true if one wishes to perform the quantum calculations in a consistent setting where back-reaction of quantum effects on spacetime geometry is calculated without the assumption of flat spacetime.

The main motivation behind this thesis was purely theoretical interest of performing quantum field theory calculations consistently in curved spacetime with a special emphasis on inflation. An almost equally important motivating factor was studying the magnitude of these effects for actual simple models and comparing these quantum corrected results to the classical predictions in the context of inflation. We believe that, at least for some models, such considerations will become relevant in the future when more accurate measurements become available. Even if in the simplest scalar field models studied here the quantum effects are by and large insignificant, this might not be the case for other models. Hence showing the theoretical path to implementing this for scalar fields provides important information for anyone seeking to perform similar calculations for more complicated models, especially when a re-summation of the quantum diagrams is used.

The framework adopted for this thesis is that of quantum field theory in curved space-time [7, 8], which corresponds to performing quantum field theory calculations in a space with classical Einsteinian gravity as a background. Consistently including gravity in our calculations would in principle mean that gravity should also be quantized, but due to the well-known complications of forming a fully quantized field theory of gravity and other interactions we decided to bypass these issues and opt for the use of classical gravity instead, at least for the time being[1]. This choice can also be motivated by the expectation that the quantum effects of gravity become significant only at a very high energy-scale. However, it may turn out that our approach fails at describing some important phenomena that result from the quantum nature of gravity. If this is true, we may still argue that viewing gravity as a classical background serves as reasonable middle ground between the classical and fully quantum approaches.

1.1 Summary of the research

The initial idea for this project was to use the standard tools of curved space field theory for inflationary calculations. The most commonly used approach for calculating and renormalizing the equations of motion for an interacting curved space quantum field theory is the Schwinger-DeWitt expansion [9], which was chosen as the method for **I**. After this calculation, the need arose to perform similar derivations, but with an expansion more suited for de Sitter space. This is something not easily incorporated in the Schwinger-DeWitt approach since it is based on an expansion around a flat spacetime. This led to a procedure that could in its entirety be done at the level of the equations of motion thus allowing an implementation of the slow-roll expansion. However, the standard curved space renormalization techniques operating at the equation of motion level are not well-suited for interacting theories. This issue was resolved in **II** by introducing a new renormalization method for curved space calculations. With the help of the slow-roll expansion and the renormalization technique of **II**, the inflationary quantum corrections were calculated in a quasi-de Sitter space in **III**. The calculation introduced an important infrared contribution that was not included in the Schwinger-DeWitt approach of **I**. It was also noticed that such a contribution might require a resumming of the loop expansion in order to regulate infrared divergent behavior. For this purpose the 2-particle-irreducible Feynman diagram expansion [10] was then implemented, a method that has not often been used for a non-static space-time.

1.2 Organization of the thesis

We will begin this thesis in chapter 2 with a brief introduction of the calculational technology and background information relevant for chapters 3 – 5. This chapter is by no means meant to be exhaustive and it is assumed that the reader is familiar with the basics of general relativity and quantum field theory as well as inflationary cosmology. Chapters 3 – 5 are organized in linear order according to the research carried out. We feel this to be the most natural choice since the topic of **II** was heavily motivated by the research done in **I** and in **III** we used the method derived in **II**. Hence, chapter 3 paraphrases the work

[1] There already exist works without this simplifying assumption. This, and other approaches are discussed in section 2.5.1.

done in **I**, chapter 4 does the same for **II** and finally chapter 5 focuses on the findings of **III**. We finish with concluding remarks in chapter 6.

Chapter 2

Basic features of curved space calculations

2.1 Classical equations of motion in curved space-time

Let us start by deriving the classical field equations of motion. Our action will consist of a scalar field φ, which couples to standard Einsteinian gravity with a Friedmann-Robertson-Walker type metric (FRW). In terms of the line element the metric can be written as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + a^2 d\mathbf{x}^2, \tag{2.1}$$

where the *scale factor* a has only dependence on time, $a \equiv a(t)$. The action includes a yet undefined potential and the standard Einstein-Hilbert action for gravity

$$\begin{aligned} S[\varphi, g^{\mu\nu}] &\equiv S_m[\varphi, g^{\mu\nu}] + S_g[g^{\mu\nu}] \\ S_m[\varphi, g^{\mu\nu}] &= -\int d^4x\sqrt{-g}\,\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi, g^{\mu\nu})\right] \qquad (2.2) \\ S_g[g^{\mu\nu}] &= \int d^4x\sqrt{-g}\,\Big[\Lambda + \alpha R\Big]. \qquad (2.3) \end{aligned}$$

According to the principle of least action, the equations of motion can be derived via variation. Varying with respect to the field we get the equation of motion for the scalar field

$$\frac{\delta S[\varphi, g^{\mu\nu}]}{\delta\varphi(x)} = 0. \tag{2.4}$$

Varying with respect to the metric yields the Einstein equation

$$\begin{aligned} \frac{2}{\sqrt{-g}}\frac{\delta S[\hat{\varphi}, g^{\mu\nu}]}{\delta g^{\mu\nu}(x)} = 0 \qquad &\Leftrightarrow \qquad \frac{2}{\sqrt{-g}}\frac{\delta S_g[\hat{\varphi}, g^{\mu\nu}]}{\delta g^{\mu\nu}(x)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m[\varphi, g^{\mu\nu}]}{\delta g^{\mu\nu}} \\ &\Leftrightarrow \qquad 2\alpha G_{\mu\nu} - \Lambda g_{\mu\nu} = T_{\mu\nu}. \qquad (2.5) \end{aligned}$$

We can write the above for the theory defined by (2.2) and (2.3) by using the expressions for the Einstein tensor in a FRW space from the formulae (A.11) and (A.12). For simplicity we are assuming no metric dependence for the potential, and we can then write the equations of motion as[1]

$$-\Box\varphi + \frac{\partial V(\varphi)}{\partial\varphi} = \ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{\partial V(\varphi)}{\partial\varphi} = 0 \tag{2.6}$$

$$3\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{M_{\rm pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] + \Lambda \tag{2.7}$$

$$-\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right] = \frac{1}{M_{\rm pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right] - \Lambda, \tag{2.8}$$

where we also used (A.7). These are the Friedmann equations [11] and in principle determine the classical dynamics of the fields, which often is assumed to be a sufficient approximation. Indeed, the will to improve upon the classical results was the main motivation for this thesis.

We can solve the acceleration $\ddot{a}$ from (2.8)

$$\frac{\ddot{a}}{a} = -\frac{1}{3}\left[\dot{\varphi}^2 - V(\varphi)\right] + \frac{\Lambda}{3}. \tag{2.9}$$

For $\varphi = 0$ and $V(0) = 0$ this gives the important special case where:

$$\frac{\ddot{a}}{a} \propto \Lambda. \tag{2.10}$$

The solution for (2.10), when supplied with (2.7), is an exponentially increasing scale factor of the form

$$a \propto e^{Ht} \tag{2.11}$$

for some constant H. This solution is called de Sitter space [12] and its accelerating behavior holds the keys to important and difficult questions in cosmology.

2.2 Inflation

Supernovae observations [13] tell us that the current universe is expanding. Considering only the observable universe, we can extrapolate backwards in time and eventually reach a state of extremely hot and dense plasma. During this hot and dense epoch the universe was filled with highly energetic particles and radiation, and was opaque to photons. Due to the expansion of space, this hot and dense plasma eventually cooled to a point where neutral atoms could form thus making the universe transparent for radiation that has been traveling freely ever since. This chain of events implies that some relic radiation should be still observable. This radiation is known as the cosmic microwave background (CMB). The CMB has most notably been measured by the COBE [14], WMAP [15] and Planck [16] missions. However, a naive interpretation of these observations also leads to severe problems. The observed CMB is extremely homogeneous and isotropic, which for the current age and expansion rate of the universe could have never been possible: the size of the region that was causally connected – and hence could reach an equilibrium – at the time when the CMB was formed is minuscule compared to the size of the horizon from which we observe it currently. This is known as the horizon problem. Another equally

[1] In order to match with standard conventions one must set $\Lambda \to -\Lambda/(8\pi G)$ and $\alpha \to 1/(16\pi G)$

important problem is the observed almost critical density of the universe. Critical density describes the density which is precisely in between an ever expanding or an eventually contracting solution: an infinitesimal increase of energy content in a universe possessing critical density would in the end reverse the expansion leading to a so-called big crunch. This reveals that the critical density is not a stable configuration and hence an initial small perturbation from the critical density will in time increase to become a large effect. The currently observed almost critical density suggests that in the past this value has to be *fine-tuned* in order to be compatible with the observed value. This is generally known as the flatness problem since a universe with critical density has no curvature i.e. is flat. A third problem of a naive extrapolation of the current scenario is that many theories predict the formation of exotic particles, such as magnetic monopoles, during the early stages of the universe. Thus far no such exotic relics have been observed.

For the reasons mentioned, it is widely accepted that the Universe at some stage went through a period of rapid, almost exponential expansion commonly known as inflation. Inflation provides natural explanations for why the universe is almost completely flat, why the CMB is so isotropic and homogeneous, and why we have not seen any exotic particles. Inflation was invented in the early eighties in [17] and [18] (see also [19]). It was realized that an early period of exponential expansion causes the size of an initially small causally connected region to increase dramatically. After a sufficiently long period of inflation the CMB observed today would have originated from a region that at one time was just one small causally connected patch of a much larger universe. During inflation we also notice the remarkable feature that the *event horizon*, i.e the physical region that may in the future causally interact with an observer, is roughly constant[2]. Since space during inflation is rapidly expanding while the physical event horizon remains constant, immediately after inflation the region inside the event horizon appears essentially flat, as long as inflation lasts long enough. Similar considerations lead to the attractive conclusion that the density of exotic particles is diluted by inflation to an unobservably small fraction of the total number.

The early models of inflation were based on the idea of the universe remaining in a metastable vacuum, where inflation ends by a phase transition. While stuck in the metastable state the potential acts as a cosmological constant as may be seen from (2.7 - 2.8). The first proposal [19] is generally categorized as "old inflation". In this model inflation ends via tunneling from the metastable state to the proper vacuum, but it turns out that this scenario is incompatible with the Universe which we observe [20], namely it suffers from the "graceful exit" problem: the tunneling operates by a process of bubble nucleation, but due to the expansion of the universe the bubble collisions do not occur sufficiently rapidly.

The "new inflation" scenario [21, 22] was devised to overcome the issues of [19]. In this proposal inflation ends not by tunneling through a barrier, but by a slow transition from the metastable state to the actual vacuum state. New inflation is still a popular model for inflation, but typically involves fine-tuning of initial conditions [23].

Most of the currently popular models fall under the banner of slow-roll inflation, where inflation includes a phase where a field slowly rolls towards a minimum of a potential and during this phase the potential acts almost as a cosmological constant. Usually the field responsible for inflation is a scalar field and is generally known as the inflaton. We can roughly categorize these models as small field and large field models, with inflationary field values smaller or larger than M_{pl}, respectively. Small field models are often motivated by

[2]This does *not* mean that regions outside the event horizon cannot have interacted in the past.

beyond standard model physics such as string theory, supersymmetry and supergravity (for examples, see [24, 25, 26]). In small field models there is the benefit that the standard tools of quantum field theory may be assumed to apply, because of the sub-Planckian field value. Unfortunately, these approaches often suffer from fine-tuning issues for the initial conditions [27]. For large field models the most popular scenario is chaotic inflation [28]. In chaotic inflation the inflationary potential is assumed to have a simple polynomial form, such as (2.38). There is no need to fine-tune the initial conditions [29], but since we are dealing with trans-Planckian field values it is not obvious what types of terms one should include in the tree-level Lagrangian. The model we are interested in this thesis belongs to the class of chaotic inflation and currently, at least for a potential dominated by a quadratic mass term, is in reasonable agreement with current data [30].

The rapid expansion of the universe is commonly assumed to evolve the universe into a non-thermal state, which lasts until the end of inflation. This means that thermal effects are relatively small during inflation. There also exist models where thermal equilibrium is maintained throughout inflation [31]. Such "warm inflation" models will not be discussed in this thesis.

When the field responsible for inflation has reached the minimum of its potential, it begins to rapidly oscillate about its equilibrium value. During this oscillatory phase the field decays into various standard model particles which, due to interactions, eventually reach thermal equilibrium. This process is generally called *reheating*[3] [32, 33]. Thus, a complete model of inflation and reheating requires fields in addition to the inflaton, but with the exception chapter 3 we will not consider such processes in this thesis.

An important prediction of many models of inflation is that the CMB will have tiny fluctuations due to quantum mechanical effects. Measurements of these CMB anisotropies are one of the best methods of verifying the predictions of inflationary models and hence provide crucial information for inflationary model building.

2.2.1 Cosmic microwave background

The cosmic microwave background is our most robust evidence for the fact that in the distant past our Universe started from a very hot and dense state. Moreover the CMB bears clear signs of the inflationary scenario. Even though the CMB is observed to be almost homogeneous, its temperature contains tiny variations which can be linked to inflation, a fact which was first showed in [21, 34, 35]. The idea is that quantum effects of the field responsible for inflation, whatever it may have been, would cause tiny fluctuations in the energy-density. These will ultimately be seen by today's observers as the CMB fluctuations. Indeed, perturbations that were originally microscopic will eventually grow into the large-scale inhomogeneities we observe today, such as planets, stars, galaxies and so forth. So according to current understanding, inflation is essential not only for the resolution of the horizon, flatness and monopole problems, it is also vital in providing the seeds for structure formation.

The cosmic microwave background anisotropies were successfully measured by a number of missions [14, 15, 16]. For our purposes the most important observable of the CMB is the amplitude of the temperature perturbations, which can be characterized by the *curvature perturbation* denoted with $\mathcal{R}$. This object essentially describes the perturbation in space, but not in time and it is precisely this quantity with which one often differentiates between various inflationary models. The standard way of deriving the prediction for $\mathcal{R}$

[3] In many standard scenarios, reheating begins with a highly non-perturbative phase dubbed *preheating*.

in a given theory is to use the free field approximation of a quantum theory of matter *and* gravity with which one may derive an equation of motion for $\mathcal{R}$ [36, 37, 38, 39][4]. The most important quantity involving $\mathcal{R}$ is the *power spectrum* $\mathcal{P}(k)$, which is defined from the two-point momentum space correlator as

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}^*_{\mathbf{k}'} \rangle \equiv (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \frac{2\pi^2}{k^3} \mathcal{P}(k). \tag{2.12}$$

Current observations have produced a number of constraints that any inflationary theory must meet and two of the most important ones are adiabaticity of the perturbations and scale invariance of the spectrum. The first one essentially means that there are no relative perturbations among the various particle species produced after inflation, i.e all particle density perturbations can be related to the same power spectrum. The nearly scale invariant behaviour of the spectrum can be written as the condition

$$\frac{d \log \mathcal{P}(k)}{d \log k} \equiv n_{\mathrm{s}} \sim 1, \tag{2.13}$$

with the current value at $n_{\mathrm{s}} = 0.9624 \pm 0.0075$ [4]. The adiabaticity condition is always satisfied for single field inflation [40], but also may be respected by multifield models if certain conditions are met [41]. The scale invariance condition is satisfied in slow-roll inflation, which we will study in the next section.

As a final note we stress that not all models of inflation predict that the spectrum of perturbations originates during inflation from the field responsible for inflation. A popular model for the spectrum is the so called curvaton scenario [42, 43, 44], where the spectrum originates from a field that is subdominant during inflation, but dominates the energy-density after inflation, thus giving rise to the observed power spectrum. In this thesis we will also briefly comment on the implications of curved space loop corrections for the curvaton scenario.

2.3 Nearly exponential inflation and the slow-roll expansion

A more detailed exposition to the slow-roll expansion can be found in [45]. In slow-roll models inflation is caused by a field slowly rolling towards a minimum of a potential, during which inflation occurs and a nearly scale invariant spectrum is formed. From (2.7) we immediately see that if

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi), \tag{2.14}$$

then we get

$$H^2 \approx \frac{V(\varphi)}{3M_{\mathrm{pl}}^2} \tag{2.15}$$

and the potential behaves nearly as a cosmological constant and we have defined the *Hubble constant* analogously to the exponential solution in (2.11)

$$\frac{\dot{a}}{a} \equiv H. \tag{2.16}$$

[4]In fact for the leading terms one may use a de Sitter space approximation for $g^{\mu\nu}$ where only matter is quantized, as is done in [38]

From (2.7) and (2.8) by using (2.14) we can write a relation for the *first Hubble slow-roll parameter*,

$$-\frac{\dot{H}}{H^2} \equiv \epsilon = \frac{\dot{\varphi}^2}{2M_{\rm pl}^2 H^2} \ll 1, \tag{2.17}$$

where the last inequality follows from the condition (2.14).

In addition to having an exponential solution of the form in (2.15) we also must require that such a condition is maintained for a sufficiently long period of time in order to have a large enough amount of inflation. We can quantify this statement by assuming that during a small time step Δt, which in this case is characterized by $1/H$ since it is the only time scale in the problem, the change in the potential is small compared to the potential itself, i.e

$$|\Delta V(\varphi)| \sim |H^{-1}\dot{V}(\varphi)| \ll V(\varphi) \quad \Leftrightarrow \quad \left|\frac{aV'(\varphi)\dot{\varphi}}{\dot{a}}\right| \ll V(\varphi). \tag{2.18}$$

We can meet this condition by postulating that the field φ has reached terminal velocity i.e. is moving at nearly constant speed so that the scalar field equation (2.6) can be written as

$$\dot{\varphi} \approx -\frac{V'(\varphi)}{3H}. \tag{2.19}$$

From the above condition we can understand the name "slow-roll" since the field has constant velocity and the kinetic energy of the field is much smaller than its potential energy. Assuming that relation (2.19) holds exactly, we can write the first slow-roll parameter as

$$\epsilon = \frac{1}{2}\left|\frac{V'(\varphi)\dot{\varphi}}{HV(\varphi)}\right| \tag{2.20}$$

and hence condition (2.18) is met by our solutions. We can quantify the approximation made in (2.19) if we define the *second Hubble slow-roll parameter*

$$\frac{\ddot{\varphi}}{\dot{\varphi}H} = \frac{\ddot{H}}{2\dot{H}H} \equiv \delta_H \ll 1. \tag{2.21}$$

It is in practice often beneficial to use another set of slow-roll parameters defined in terms of the potential alone. We can write the first potential slow-roll parameter by using in (2.20) again (2.14) and (2.19) giving

$$\epsilon_V = \frac{M_{\rm pl}^2}{2}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \ll 1. \tag{2.22}$$

In order to derive the second potential slow-roll parameter, we can take a time derivative of the condition (2.19) and use the first potential slow-roll parameter to deduce the relation

$$\delta_V = M_{\rm pl}^2\frac{V''(\varphi)}{V(\varphi)} \ll 1. \tag{2.23}$$

It is important to realize that in deriving the potential slow-roll parameters we must assume that (2.19) holds and hence from the smallness of (2.22) and (2.23) alone the desired form of the solutions does not follow.

Using these parameters one can efficiently expand the equations of motion, which provides an indispensable tool for solving and analyzing inflationary dynamics. It is quite often useful to rewrite the two Friedmann equations (2.7) and (2.8) as an equation for H and the first Hubble slow-roll parameter ϵ

$$3H^2 = \frac{1}{M_{\rm pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] + \Lambda \tag{2.24}$$

$$2\epsilon H^2 = \frac{\dot{\varphi}^2}{M_{\rm pl}^2} = \frac{1}{M_{\rm pl}^2}\left[T_{00} + \frac{T_{ii}}{a^2}\right], \tag{2.25}$$

for a potential with no dependence on the metric. In the above equations the second one can be viewed as the dynamical one, i.e the one that is responsible for the time evolution and the first one only fixes the initial conditions.

The slow-roll expansion parameters can also handily be used to express important relations. Assuming roughly exponential inflation we can define the number of *e-folds* corresponding to a value for the scale factor a_0 as

$$N \equiv \log\left[\frac{a(t)}{a(t_0)}\right], \tag{2.26}$$

which can be written in terms of the slow-roll parameters as a function of the field values for φ

$$N \approx \int_{\varphi}^{\varphi_0} \frac{d\varphi/M_{\rm pl}}{\sqrt{2\epsilon_V}}. \tag{2.27}$$

It is generally assumed that one requires around 60 e-folds of inflation to resolve the horizon problem [46]. Similarly, for the spectral index (2.13) we may write [38]

$$n_{\rm s} = 1 + 2\delta_V - 6\epsilon_V, \tag{2.28}$$

from which it is apparent that approximate scale invariance of the spectrum is a natural prediction of slow-roll inflation. We will make extensive use of the slow-roll expansion in the quantum setting in chapter 5.

2.4 Inclusion of quantum effects

In principle it is known how to promote a classical field into a quantum object and write the equations of the previous section in the quantum setting. If we have a theory which is expressed with a generic field variable ψ, which is not necessarily a scalar, in standard field quantization we promote it into an operator denoted as $\hat{\psi}$ possessing certain commutation relations. The measurable quantities in this context are expectation values, which can be expressed via the generating functional as

$$\langle\psi(x_1)\psi(x_2)\cdots\rangle = \left(\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\cdots\right)Z[J]\bigg|_{J=0}. \tag{2.29}$$

In the Feynman path integral approach the generating functional has the representation

$$Z[J] = \int \mathcal{D}\psi\; e^{iS[\psi]+i\int d^4x J\psi}. \tag{2.30}$$

In practice it is impossible to calculate analytic expressions for the correlators without making use of approximate methods, at least for the theories we are interested in, and we will use standard perturbative approximations. In particular, in this thesis we will make use of the loop expansion to first order, with the exception of chapter 5.

Performing the loop expansion is a standard calculation [47], which we now show for the action defined in (2.2). We start by quantizing the scalar field variable φ and defining the fluctuation operator as $\hat{\varphi} \to \varphi + \hat{\phi}$, where we used a simplified notation for the expectation value $\langle\hat{\varphi}\rangle \equiv \varphi$. Next we expand (2.2) around $\hat{\phi} = 0$ giving to quadratic order[5]

$$S_m[\varphi, \hat{\phi}, g^{\mu\nu}] = -\int d^n x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi, g^{\mu\nu})\right] - \frac{1}{2}\int d^n x\sqrt{-g}\,\hat{\phi}\Big[-\Box + M^2\Big]\hat{\phi} + \cdots, \tag{2.31}$$

where we have defined the effective mass

$$M^2 \equiv \frac{\partial^2 V(\varphi, g^{\mu\nu})}{\partial\varphi^2}. \tag{2.32}$$

The effective mass is an essential concept when using a one-loop approximation. From the expansion (2.31), we can write an equation of motion for the fluctuation operator

$$\Big[-\Box + M^2\Big]\hat{\phi} = 0, \tag{2.33}$$

which can be expanded via the creation and annihilation operators

$$\hat{\phi} = \int d^{n-1}k\big[a_{\mathbf{k}}u_{\mathbf{k}} + a^*_{\mathbf{k}}u^*_{\mathbf{k}}\big], \tag{2.34}$$

with the standard commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta^{n-1}(\mathbf{k} - \mathbf{k}'). \tag{2.35}$$

When applying perturbative quantum field theory the core object around which the calculation is based is the propagator, which can be expressed via the fluctuation operator and the time ordering operator $\hat{T}$ as

$$G(x, x') = \langle 0|\hat{T}\big\{\hat{\phi}(x)\hat{\phi}(x')\big\}|0\rangle, \tag{2.36}$$

where $|0\rangle$ is a state annihilated by $\hat{a}_{\mathbf{k}}$ from (2.34). This shows the important role of the effective mass in the one-loop approximation that the entire field dependence of the quantum loops is given by the effective mass.

The equation of motion for φ, referred to as the field equation of motion, can also be derived from (2.31) and in comparison to (2.6) now includes an important quantum term

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + \frac{\partial V(\varphi, g^{\mu\nu})}{\partial\varphi} + \frac{1}{2}\frac{\partial^3 V(\varphi, g^{\mu\nu})}{\partial\varphi^3}\langle\hat{\phi}^2\rangle = 0, \tag{2.37}$$

which for example for a theory with

$$V(\varphi, g^{\mu\nu}) = \frac{1}{2}m_0^2\varphi^2 + \frac{1}{2}\xi_0 R\varphi^2 + \frac{\lambda_0}{4!}\varphi^4, \tag{2.38}$$

[5] In the one-loop approximation, the terms linear in $\hat{\phi}$ can be discarded. This can be seen by using the classical equation of motion and discarding higher loop effects.

which will be the choice for our calculations in chapters 4 – 5, gives.

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + m_0^2\varphi + \xi_0 R\varphi + \frac{\lambda_0}{3!}\varphi^3 + \frac{\lambda_0}{2}\varphi\langle\hat{\phi}^2\rangle = 0. \tag{2.39}$$

Suppose for a moment that one has a solution for $\hat{\phi}$ and also that the behavior of the scale factor a as a function of time is known. There is then one more step before we can derive solutions for φ from the equation of motion in (2.39). A generic feature of quantum field theories is that initially most correlation functions, $\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\cdots\rangle$, are infinite. The process of removing these divergences, i.e. renormalization[6] is known for most of the standard field theories in Minkowski space and its implementation is straightforward, although often requires tedious calculations. In order for this procedure to follow through, we must require that a redefinition of the constants introduced by the original action is enough to cancel all the appearing divergences to all orders in the perturbative expansion. A theory with this property is generally called renormalizable. The most popular renormalization method is to introduce a counter term for each parameter of the original action and then tune these in such a way that the divergences are canceled. The practical implementation of the renormalization procedure requires one to first modify the theory in such a way that the infinities are transformed into numbers, so that standard algebra may be used. This step is known as regularization. We will here implement dimensional regularization [48], where we analytically continue our spacetime from 4 dimensions to n, which successfully removes the divergent behavior.

In this thesis the inclusion of counter terms will be denoted by writing each constant of the classical action with a subscript "0". So a generic constant c_0 will include a finite physical contribution and an infinite counter term as

$$c_0 = c + \delta c, \tag{2.40}$$

where δc signifies the counter term. If, for simplicity, we neglect the counter term for the kinetic term $\Box$, we can write (2.39) with the prescription (2.40) as

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} + m^2\varphi + \xi R\varphi + \frac{\lambda}{3!}\varphi^3 + \delta m^2\varphi + \delta\xi R\varphi + \frac{\delta\lambda}{3!}\varphi^3 + \frac{\lambda}{2}\varphi\langle\hat{\phi}^2\rangle = 0. \tag{2.41}$$

Should it occur that the counter terms introduced by the classical action are not enough for cancelling the quantum infinities, then the theory has little predictive power, at least in the perturbative sense. This is because at each order in the loop expansion one must introduce additional experimentally determined constants, a process which will continue *ad infinitum*. In the above case this means that δm^2, $\delta\xi$ and $\delta\lambda$ must cancel the infinities introduced by $\langle\hat{\phi}^2\rangle$. In the one-loop approximation renormalizability requires that in (2.31) the first line, which can be considered zeroth order or classical, the constants include counter terms, but in the second line there are no counter terms. This is because it is already of one-loop order and a counter terms times a one-loop term is effectively a two-loop correction and hence beyond the one-loop approximation, which is visible in (2.41) having no term $\propto \delta\lambda\langle\hat{\phi}^2\rangle$.

One of the most important consequences of renormalization is that the physical parameters of the theory, such as m and λ, may be viewed to have a dependence on the energy scale. The exact form of this dependence is specific to the particular theory in question

[6]In fact, even for a completely finite theory some kind normalization of quantities would still be required.

and may lead to surprising and important consequences, such as an asymptotically free theory at high energy limit in the case non-Abelian gauge theory [49]. A transformation between various energy scales at which the parameters of the theory are defined is called a *renormalization group* transformation[7], which provides a useful tool for field theory. For example, it can be used as a means of improving the perturbative expansion [50].

What we so far have not discussed is that one also gets quantum corrections to the Friedmann equations (2.7) and (2.8), and it is not at all trivial that the renormalization procedure can be implemented for the energy-momentum. A related matter is that we have now merely quantized the field φ, but a completely consistent approach would also include a quantum theory of gravity. Unfortunately, no such theory exists. The fundamental reason behind this issue lies in the lack of consistent perturbative renormalizability of quantized Einsteinian gravity, shown to one-loop order in [51]. This is not to say that at the moment it is not possible in some form to include effects of quantum gravity the calculations and in fact several works already exist where these effects have been considered in the context of inflation. We will briefly return to this issue in section 2.5.1.

As a first approximation one could calculate the quantum corrections in flat spacetime where the renormalization procedure and solution for the mode equation are known and in general the whole procedure is straightforward. This approach suffers from some inconsistencies, since it completely neglects the gravitational effects for the quantum fluctuations but nevertheless can be viewed as the first approximation for the inclusion of quantum effects. A step closer to a complete quantum formulation would be to assume that gravity is classical, but the quantum effects take place in the presence of classical gravity. In this approach there is again no need to worry about quantizing the metric, but renormalization becomes a non-trivial issue, since the quantum divergences back-react on classical gravity. Fortunately, consistent renormalization is possible in this approach [7]: it turns out that with the addition of new terms in the gravity Lagrangian in (2.3) all divergences can be consistently removed. This construction is often called *quantum field theory in curved spacetime* or *curved space quantum field theory*. This will be the framework for our calculations.

As a practical point, so far we have assumed that we were able to solve the mode equation in (2.33) for some given $g^{\mu\nu}$. In principle this equation is coupled to the quantum corrected versions of (2.6 – 2.8) forming a highly non-linear set of equations, especially if one wishes to include gravity in the quantum dynamics. Indeed, even for simple interacting theories, the effective mass in (2.33) has a dependence on the field expectation value φ, which in general is not a constant. Similarly, the derivative term $\Box$ introduces additional dependencies to $g^{\mu\nu}$. It is often very challenging to solve the mode equation (2.33) and finding the approximation suited for ones purposes usually forms the core of the problem.

When using quantum field theory in curved spacetime, there are roughly two paths to the quantum corrected versions of the equations (2.6 – 2.8): The first is to derive a so called effective action [47], usually denoted as $\Gamma[\varphi, g^{\mu\nu}]$, which gives the quantum corrected equations of motion by variation just like the classical action in (2.4) and in (2.5) i.e.

$$\frac{\delta\Gamma[\varphi, g^{\mu\nu}]}{\delta g^{\mu\nu}} = 0, \quad \frac{\delta\Gamma[\varphi, g^{\mu\nu}]}{\delta\varphi} = 0, \tag{2.42}$$

with the first being the Einstein equation and the second the equation of motion of the field. Here it must be borne in mind that now φ represents the expectation value of the field, $\langle\hat{\varphi}\rangle \equiv \varphi$. The second way would be to vary the quantized action $S[\hat{\varphi}, g^{\mu\nu}]$ with

[7]Formally the above mentioned operations do not form a group [47].

respect to the metric $g^{\mu\nu}$ and the operator $\hat{\varphi}$ and only afterward calculate the expectation value as[8]

$$\left\langle \frac{\delta S[\hat{\varphi}, g^{\mu\nu}]}{\delta \hat{\varphi}} \right\rangle = 0, \quad \left\langle \frac{\delta S[\hat{\varphi}, g^{\mu\nu}]}{\delta g^{\mu\nu}} \right\rangle = 0. \tag{2.43}$$

The first of these approaches is implemented in chapter 3 and the latter in chapters 4 and 5.

2.5 Quantum field theory in curved spacetime

Quantum field theory in curved spacetime in general means a prescription with quantum fields with a classical curved background [7, 8]. This means that no fluctuations of the metric are considered, which in turn for the functional integral approach means that the generating functional $Z[J]$ has path integration over only the matter fields. For a theory with a single field φ and a classical matter action as in (2.2) we can write the generating functional (2.30) as

$$Z[J] = \int \mathcal{D}\varphi \; e^{iS[\varphi, g^{\mu\nu}] + i\int d^4x\sqrt{-g}\; J\varphi}, \tag{2.44}$$

where the action has a matter part and a gravitational part

$$S[\varphi, g^{\mu\nu}] \equiv S_m[\varphi, g^{\mu\nu}] + S_g[g^{\mu\nu}], \tag{2.45}$$

with

$$S_m[\varphi, g^{\mu\nu}] = -\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi, g^{\mu\nu})\right] \tag{2.46}$$

$$S_g[g^{\mu\nu}] = \int d^4x\sqrt{-g}\left[\Lambda_0 + \alpha_0 R + \beta_0 R^2 + \epsilon_{1,0}R_{\alpha\beta}R^{\alpha\beta} + \epsilon_{2,0}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right]. \tag{2.47}$$

As explained in section 2.2 there are many choices for a tree-level inflationary potential, even for models with just one scalar field. The models studied in this thesis belong to the class of chaotic models introduced in [28] with a single scalar field. In order to encompass the most popular chaotic models with only a quadratic or a quartic potential with possible non-minimal coupling to gravity, the choice for our tree-level potential is

$$V(\varphi, g^{\mu\nu}) = \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\xi R\varphi^2 + \frac{\lambda}{4!}\varphi^4. \tag{2.48}$$

In chapter 3 where we study a model with two fields φ and σ, where, in addition to the above, we include also an interaction term between the two fields proportional to $\sigma^2\phi^2$. These choices correspond to a renormalizable theory and thus all of our models may be studied via curved space field theory.

In comparison to classical field theory defined by the action (2.2) and (2.3) there is now a major difference: in the gravity contribution for the action we have introduced higher order tensors, which are needed for consistent renormalization of the theory [7].[9]

[8] Mathematically a more concise way of deriving the field equation is to first vary with respect to φ and then quantize the resulting equation.

[9] We assume that we are in an unbounded space and hence one can leave out terms that are total derivatives. For a more general action with out this requirement see **I**.

Our assumption is that these terms are only needed for renormalization i.e. the physical coupling constants for these higher order terms are negligible

$$\beta_0 = 0 + \delta\beta \tag{2.49}$$

and similarly for the constants $\epsilon_{1,0}$ and $\epsilon_{2,0}$. So even when keeping gravity as a purely classical field, quantum corrections generate non-Einsteinian interactions which have to be included for the consistency of the results. For example when deriving the effective action in section 3, we see that there is no way of removing certain infinities if such terms are not included.

The energy-momentum tensor, which now includes quantum corrections, is still defined via variation as in (2.5). If we calculate the energy-momentum from a renormalized effective action as in (2.42) we can simply write

$$2\alpha G_{\mu\nu} - \Lambda g_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle \quad \Leftrightarrow \quad -\frac{2}{\sqrt{-g}} \frac{\delta\Gamma[\varphi, g^{\mu\nu}]}{\delta g^{\mu\nu}} = 0. \tag{2.50}$$

In this approach the difficult part of the calculation lies in finding an expression for $\Gamma[\varphi, g^{\mu\nu}]$. If, on the other hand we derive the Einstein equation without performing any renormalization, we can do it simply by taking expectation values of the varied action as in (2.43), giving us

$$2\alpha G_{\mu\nu} - \Lambda g_{\mu\nu} = \langle \hat{T}_{\mu\nu} \rangle \equiv -\frac{2}{\sqrt{-g}} \left\langle \frac{\delta S_m[\hat{\varphi}, g^{\mu\nu}]}{\delta g^{\mu\nu}} \right\rangle - \frac{2}{\sqrt{-g}} \frac{\delta S_{\delta g}[g^{\mu\nu}]}{\delta g^{\mu\nu}}. \tag{2.51}$$

The equation now has a divergent quantum piece and counter terms from the matter action (included in S_m) and a gravitational counter term contribution coming from the generalized gravitational action in (2.47). The gravitational counter terms can be expressed with the tensors from (A.1 – A.5) as

$$\frac{2}{\sqrt{-g}} \frac{\delta S_{\delta g}[g^{\mu\nu}]}{\delta g^{\mu\nu}} = -g_{\mu\nu}\delta\Lambda + 2\delta\alpha G_{\mu\nu} + 2\delta\beta \ {}^{(1)}H_{\mu\nu} + 2\delta\epsilon_1 \ {}^{(2)}H_{\mu\nu} + 2\delta\epsilon_2 H_{\mu\nu} \equiv \delta T^g_{\mu\nu}. \tag{2.52}$$

In the one-loop approximation, which is used in this thesis throughout except in chapter 5, we can conveniently split the energy-momentum tensor into classical, quantum and counter term parts respectively as

$$\begin{aligned} \langle \hat{T}_{\mu\nu} \rangle &\equiv T^C_{\mu\nu} + \langle \hat{T}^Q_{\mu\nu} \rangle + \delta T^m_{\mu\nu} - \delta T^g_{\mu\nu} \\ &\equiv T^C_{\mu\nu} + \langle \underline{\hat{T}^Q_{\mu\nu}} \rangle, \end{aligned} \tag{2.53}$$

where we have introduced the underline to symbolize a finite quantum contribution. Since the gravitational counter term includes variations of the higher order tensors, e.g R^2, $R_{\mu\nu}R^{\mu\nu}$ coming from (2.47) one might wonder whether these higher order contributions would also introduce extra degrees of freedom, since in higher order tensors one has third and fourth derivatives of the scale factor a. This would require one to impose more boundary conditions than in the classical scenario. However, as was shown in [52] the third and fourth order derivative terms in the equations of motion can be expressed with $\ddot{a}$ and $\dot{a}$ to any order in perturbation theory.

After these introductory remarks we are now ready to proceed to discuss the work done in **I**, **II** and **III**, but first we will briefly review work that is complementary to that of ours.

2.5.1 Related models and approaches

Inflationary quantum corrections have been previously calculated for many models using a variety of techniques. Below we list some of the relevant studies and note that due to the large volume of work in this field it is virtually impossible to present an exhaustive list.

A popular model slightly different from what were are interested in here and where traditionally quantum corrected effective equations have played a significant role, is where inflation is caused by the standard model Higgs particle [53]. This is because the couplings of the model are fixed to be the standard model ones, and one must carefully analyze their running behaviour in order to deduce the respective sizes at the scale of inflation. In this framework the Lagrangian is essentially of the form (2.48), with a non-minimal coupling $\xi \sim 10^5$ in order to find agreement with current observations. The inclusion of quantum corrections usually proceeds in a slightly different manner compared to us because of the large non-minimal coupling. Relevant works include [54, 55, 56, 57, 58, 59, 60, 61, 62], where, with the exception of [61, 62], the quantum corrected effective equations were calculated in flat spacetime. For Higgs inflation an expansion in terms of the slow-roll parameters of section (2.3) is questionable, again because of the largeness of ξ[10].

Another inflationary model sometimes studied using (nonequilibrium) field theory in a curved background is "new inflation". For example, in [63, 64, 65, 66] the inflationary quantum corrections are calculated consistently in a curved background, including back-reaction of the quantum dynamics on the gravitational field, with the exception of [63]. For a related use of nonequilibrium techniques, see [67]. Since new inflation is assumed to start in a thermal equilibrium state and inflation is driven by vacuum energy, the initial conditions and hence the conclusions in this setting differ from those from our studies.

There are of course other approaches to inflationary quantum corrections than our method of using curved space field theory. The fact that we have included no fluctuations of gravity is a choice that is well-motivated by the desire to obtain a renormalizable theory, but significant steps have already been taken in terms of including also the gravity fluctuations. Ever since the classic paper [51], there has been much interest in quantum effects of gravity. For inflation, they have been studied for example in [68, 69, 70, 71, 72, 73, 74, 75]. In this approach one necessarily encounters the non-renormalizability of gravity and the conceptual problems it poses.

Another method for studying inflationary quantum corrections is the stochastic quantization approach [76, 77, 78, 79, 80, 81]. In the stochastic approach one divides the dynamics of the field into a long wave-length part that is treated as a classical (but stochastic) variable and a small wave-length part where the quantum properties are maintained. With this approximation, one may write the quantum corrected field equation of motion as a Langevin-type equation with a Gaussian random noise representing the quantum effects. It may be argued that the stochastic approach gives very similar results to a full quantum approach and recently this view was supported by [82] where it was discovered that to two-loop order stochastic quantization gives identical results to a field theory calculation for the infrared part of the two-point function.

Renormalization group methods have also been used in the cosmological context [83, 84, 85, 86, 87, 88]. It has been shown that the running of constants, and especially of the

[10] Because of this fact it is often argued that before the quantum effects may be calculated one should perform a Weyl scaling on the metric, $g^{\mu\nu} \to \Omega^2 g^{\mu\nu}$ in order to remove the non-minimal term from the Lagrangian.

cosmological constant, potentially leads to important effects, for example that an epoch of inflation can solely be caused by a running cosmological constant. Recently, it was shown by using nonperturbative renormalization group techniques [89] that quantum corrections restore classically broken symmetries in a n dimensional de Sitter space with scalar fields [90].

Additionally, we should stress that in our approximation the quantum corrections enter only through the effective equations of motion. This means that the expression for the power spectrum (2.12) or the spectral index (2.28) is the canonical one that can be found from standard literature, e.g. [36, 38]. However, after the work presented in [91, 92] there has been increasing interest in calculations where loop corrections are calculated for the power spectrum and other n-point correlators of $\mathcal{R}_{\mathbf{k}}$. Recently they have been studied by a number of authors [93, 94, 95, 96, 97, 98, 99]. In this approach there are still some open questions concerning infrared divergences and secularity [100]. As it happens, the calculation of **III** gives precisely an example of how re-summing loop diagrams may cure infrared divergences at the one-loop order and this fact leads us to believe that the calculations presented there potentially provide a novel angle on the problem. Some comments on this matter will be given in the concluding section of this thesis.

Chapter 3

Effective action in curved spacetime

The effective action formalism has for a long time been a standard part of the particle physicists' calculational techniques. It was used most notably in [50], where it was shown that quantum corrections may significantly alter the naive classical predictions. The effective action provides a systematic method for calculating the quantum corrections to the classical equations of motion and properly renormalizing the result, so a priori it seems well-suited for our purposes. Unfortunately, the most uses of this approach have been in Minkowski space applications and when one wishes to include spacetime curvature, generalizations of the flat space techniques are needed. In curved spacetime the action's dependence on the metric $g^{\mu\nu}$ makes explicit calculations highly complicated.

Probably the most widely used method for calculating the effective action in curved space is a gradient expansion, commonly known as the Schwinger-DeWitt expansion [101, 102]. This method was used in the curved space setting in for example [103, 104, 105, 106, 107, 108, 109, 110, 111]. With this approach one may calculate the result in principle to as high an order as one pleases, but only the first few orders are soluble in practice [112]. In our calculation we truncated the expansion at the second order, which is where the last divergence occurs. This means that our renormalized result contains all the important logarithmic running terms.

In this calculation the only approximation made is that fields and their derivatives are small with respect to the effective mass, indicating the possibility of applying the results to problems outside the context of inflation and possibly even outside cosmology altogether.

We chose to implement the Schwinger-DeWitt procedure for a model of two scalar fields that couple to one another, in addition to having mass and self coupling terms. This way our solutions include two particle models.

Our aim in this chapter is to show how to derive the effective action and analyze the results. The quantum corrected equations of motion will then follow by variation just like for a classical action as in (2.42), where again we emphasize that φ now represents the expectation value of the field. It is a simple calculation to show that the effective action can be derived via a functional Legendre transformation of the generating functional with respect to the source J,

$$\Gamma[\varphi, g^{\mu\nu}] \equiv \int d^4x\sqrt{-g}\;\mathcal{L}_{eff}[\varphi, g^{\mu\nu}] \equiv -i\log Z[J] - \int d^4x\sqrt{-g}\;J\varphi, \tag{3.1}$$

which can be proven by operating on the right hand side of (3.1) with $\delta/(\delta\varphi)$. Since we have managed to express the effective action with the generating functional (2.30), we can use standard loop expansion as in (2.31) in order to find an explicit expression. An effective action formed in the above manner can be shown the consist of only Feynman graphs that are *one-particle-irreducible* [47], which means that they cannot be made disconnected by cutting a single line. For this reason it is often referred to as 1PI effective action. This alone still does not provide us with enough simplification in order to calculate an explicit result in curved space for $\Gamma[\varphi, g^{\mu\nu}]$. This is mostly due to the arbitrariness of $g^{\mu\nu}$. Because of this we will next use the Schwinger-DeWitt expansion technique for finding an approximation for the one-loop result to $\Gamma[\varphi, g^{\mu\nu}]$.

3.1 Schwinger-DeWitt expansion

We now show the steps for finding an expression for the effective action via the Schwinger-DeWitt expansion. We start from (3.1) by using the definitions for the generating functional (2.30) and the 1-loop expansion for the action from (2.31), which allow us to write the effective action to 1-loop order as

$$\Gamma[\varphi, g^{\mu\nu}] = \int d^4x\sqrt{-g}\mathcal{L}_{eff} = \Gamma^{(0)}[\varphi, g^{\mu\nu}] + \Gamma^{(1)}[\varphi, g^{\mu\nu}] + \cdots , \tag{3.2}$$

with

$$\Gamma^{(0)}[\varphi, g^{\mu\nu}] = S[\varphi, g^{\mu\nu}]_0, \quad \Gamma^{(1)}[\varphi, g^{\mu\nu}] = -\frac{i}{2}\text{Tr}\ \log\, G(x, x'), \tag{3.3}$$

where the subscript "0" signifies that all the constants are considered bare and can be split into a finite part and a divergent counter term as in (2.40). We also used the symbolic notation for the functional determinant

$$\frac{1}{\sqrt{\det M}} = \int \mathcal{D}\varphi\ e^{-\frac{1}{2}\varphi M\varphi}, \tag{3.4}$$

the formula

$$\det M = e^{\text{Tr}\log M} \tag{3.5}$$

and the fact that the propagator can be derived by inverting the equation

$$\Big[- \Box_x + M^2 \Big] G(x, x') = -i\frac{\delta(x - x')}{\sqrt{-g}}. \tag{3.6}$$

The above formula can be proven by operating with $-\Box_x + M^2$ on the propagator definition (2.36) and using the commutation relation for the field $\hat{\phi}$ and its momentum conjugate $\hat{\pi} = \dot{\hat{\phi}}$

$$\big[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})\big] = i\delta^{(n-1)}(\mathbf{x} - \mathbf{y}). \tag{3.7}$$

So if we can find an expression for the trace logarithm of the propagator in (3.3), we have our result for the 1-loop the effective action.

One way of finding an expression for $\Gamma^{(1)}[\varphi, g^{\mu\nu}]$, is to use the Schwinger-DeWitt expansion, otherwise known as the heat kernel method, introduced for curved spacetime in [113] (see references for other uses). We must first write the trace of a logarithm in (3.3) as a proper-time integral over a yet undefined kernel function K

$$\frac{i}{2}\text{Tr}\ \log\, G^{-1}(x, x') = -\frac{i}{2}\mu^{4-n} \int d^n x\sqrt{-g} \int_0^\infty \frac{d\tau}{\tau} K(\tau; x, x). \tag{3.8}$$

Because of the divergent behaviour that occurs in four dimensions for $\Gamma^{(1)}[\varphi, g^{\mu\nu}]$, we have dimensionally regularized the above integral to have the dimension $n = 4 - \epsilon$, as discussed in section 2.4. We have also added an arbitrary scale μ in order to maintain the proper dimension of the action. In the appendixes of **I** one may find the Schwinder-DeWitt method in explicit detail, but for the purpose of this text we simply state the result, which is

$$K(\tau; x, x) = i\frac{\Omega(\tau; x, x)e^{-iM^2_{\mathbf{SD}}\tau}}{(4\pi i\tau)^{n/2}}, \tag{3.9}$$

where $M_{\mathbf{SD}}$ is an effective mass parameter that is *different* from the definition in (2.32)

$$M^2_{\mathbf{SD}} = M^2 - \frac{R}{6} \tag{3.10}$$

and the Ω has a small proper-time expansion

$$\Omega(\tau; x, x) = \sum_{k=0}^{\infty} a_k(x, x)(\tau i)^k. \tag{3.11}$$

All the essential physical information is now contained in the expansion coefficients $a_k(x, x)$. These coefficients have been known for many years [101, 102], and results for the first four can be found for example in [112]. In our calculation we will truncate the expansion after the third $a_k(x, x)$. Explicitly the needed coefficients are then

$$\begin{aligned}
a_0(x, x) &= 1, \\
a_1(x, x) &= 0, \\
a_2(x, x) &= -\frac{1}{6}\Box M^2_{\mathbf{SD}} + \frac{1}{180}(\Box R + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta})
\end{aligned} \tag{3.12}$$

Here we would also like to point out that the above procedure is not restricted to the case of real scalar fields, but can be equally well applied to a large class of operators. This includes fields of higher spin, gauge theories and quantum gravity, see [114] for a detailed account. The result for the effective action can now be written as

$$\Gamma^{(1)}[\varphi, g^{\mu\nu}] = \int d^n x\sqrt{-g}\, \frac{1}{2(4\pi)^{n/2}}\left(\frac{M_{\mathbf{SD}}}{\mu}\right)^{n-4} \sum_{k=0}^{\infty} M^{4-2k}_{\mathbf{SD}} a_k(x, x)\Gamma(k - n/2). \tag{3.13}$$

From the argument of the gamma function, we see that at the $n = 4$ limit, we have divergent behavior for the first three terms. From the expressions in (3.12) we see that we have divergences multiplying not just R but also the higher order tensors such as R^2 and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. This is the reason why one is forced to introduce the non-Einsteinian tensors in the gravity action in (2.47): without them, we do not have the necessary counter terms. We also see that in order for the expansion to be sensible, the effective mass $M_{\mathbf{SD}}$ must be much larger than the coefficients $a_k(x, x)$, which can be shown to consist of an increasing number of derivatives of the matter fields and the gravitational tensors [9].

This derivation seems suspiciously simple, after all we are doing quantum field theory consistently in a curved background. The steps shown here are of course not the whole story and here we have left out precisely the non-trivial parts of the calculation, namely the derivation for the coefficients $a_k(x, x)$. The great power of the Schwinger-DeWitt expansion lies precisely in the fact that most of the steps need not be repeated, but one may simply implement the already existing – and very general – results for the scenario of particular interest.

3.2 Renormalization of the effective action

Before we can write down a finite, well-defined expression for our effective action, we must first use a consistent renormalization procedure for removing the divergences in (3.13). Since we have already dimensionally continued our spacetime to n dimensions, we can algebraically manipulate the expression (3.13) in such a way that all infinities, which for the dimensionally regularized result appear as terms with $(n-4)^{-1}$ type poles, are removed from the result by tuning the counter terms introduced by $S[\varphi, g^{\mu\nu}]_0$ in (3.3). After this subtraction we may formally take the limit $n \to 4$. The need to remove the divergences still does not completely fix the procedure, since to a counter term δc one could in principle add any finite portion and still obtain a result that is perfectly finite. The crucial difference between various *subtraction schemes*, which are separated from one another by different finite parts of the counter terms, is that they ultimately define different physical constants for the theory. The objective of course is that the constants of the quantum theory are such that their physical interpretation coincides with the classical constants as much as possible, for the scenario of interest. The method adopted here is that for some scales $\psi_i = \mu_i$[1] we match each constant to equal the classical result. When written explicitly for a single scalar field model, this means that the renormalization conditions are[2]

$$
\begin{aligned}
&\left.\frac{\partial^2 \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial \dot{\varphi}^2}\right|_{\mu_i} = \left.\frac{\partial^2 \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial \dot{\varphi}^2}\right|_{\mu_i}, \qquad &&\left.\frac{\partial^2 \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial \varphi^2}\right|_{\mu_i} = \left.\frac{\partial^2 \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial \varphi^2}\right|_{\mu_i}, \\
&\left.\frac{\partial^4 \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial \varphi^4}\right|_{\mu_i} = \left.\frac{\partial^4 \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial \varphi^4}\right|_{\mu_i}, &&\left.\frac{\partial^3 \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial \varphi^2 \partial R}\right|_{\mu_i} = \left.\frac{\partial^3 \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial \varphi^2 \partial R}\right|_{\mu_i}, \\
&\mathcal{L}_{eff}[\varphi, g^{\mu\nu}]\big|_{\mu_i} = \mathcal{L}[\varphi, g^{\mu\nu}]\big|_{\mu_i}, &&\left.\frac{\partial \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial R}\right|_{\mu_i} = \left.\frac{\partial \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial R}\right|_{\mu_i}, \\
&\left.\frac{\partial^2 \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial R^2}\right|_{\mu_i} = \left.\frac{\partial^2 \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial R^2}\right|_{\mu_i}, &&\left.\frac{\partial \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial R_{\alpha\beta} R^{\alpha\beta}}\right|_{\mu_i} = \left.\frac{\partial \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial R_{\alpha\beta} R^{\alpha\beta}}\right|_{\mu_i}, \\
&\left.\frac{\partial \mathcal{L}_{eff}[\varphi, g^{\mu\nu}]}{\partial R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}\right|_{\mu_i} = \left.\frac{\partial \mathcal{L}[\varphi, g^{\mu\nu}]}{\partial R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}\right|_{\mu_i}.
\end{aligned}
\tag{3.14}
$$

One usually imposes the requirement that the scales μ_i are constrained to form a proper solution of the equations of motion: they are not completely arbitrary. After deriving expressions for the counter terms via the conditions (3.14), we can write down a well-defined expression for the effective action (3.2). Next we may proceed to study the explicit results.

3.3 Some results for a two scalar field model

Implementing the Schwinger-DeWitt expansion for a model with more than one scalar field comes as a natural generalization from the discussion in section 3.1. The only complication that might arise is that if the action has terms that couple different fields together, one must first disentangle these mixing terms so that the result can be written as a sum of various trace logarithms. In this manner the Schwinger-Dewitt expansion can be used for each contribution separately. This process comes about via diagonalizing the action,

[1] $\psi_1 = \dot{\varphi}$, $\psi_2 = \varphi$, $\psi_3 = R$, $\psi_4 = R_{\alpha\beta}R^{\alpha\beta}$ and $\psi_5 = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$.

[2] One may also include linear and trilinear conditions.

which is a question of simple linear algebra. We performed the above analysis for a model with the following matter action

$$\begin{aligned} S_m[\varphi, \sigma, g^{\mu\nu}] \quad \equiv \quad & \int d^4x\sqrt{-g}\,\Big[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \eta_\varphi\Box\varphi^2 - \frac{m_\varphi^2}{2}\varphi^2 - \frac{1}{2}\xi_\varphi R\varphi^2 \\ & -\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma + \eta_\sigma\Box\sigma^2 - \frac{m_\sigma^2}{2}\sigma^2 - \frac{1}{2}\xi_\sigma R\sigma^2 \\ & -\frac{g\varphi^2\sigma^2}{4} - \frac{\lambda_\sigma\sigma^4}{4!} - \frac{\lambda_\varphi\varphi^4}{4!}\Big]. \end{aligned} \tag{3.15}$$

In **I** we have a fully general result for the effective action for the theory in (3.15), but the numerical studies were done for an unbounded space – which means that total derivatives vanish – and with the choices

$$\xi_\sigma = \xi_\varphi = \lambda_\sigma = \lambda_\varphi = 0 \tag{3.16}$$

and furthermore assuming that only one of the fields develops an expectation value, i.e,

$$\varphi = 0. \tag{3.17}$$

The renormalization scale was chosen as zero for all matter fields and Minkowski space for the metric. This means that the constants of the theory correspond to the classical ones at the point $\varphi = \sigma = 0$ and $g^{\mu\nu} = \eta^{\mu\nu}$. This gives the effective Lagrangian[3]

$$\begin{aligned} \mathcal{L}_{eff} \quad = \quad & -\frac{\partial_\mu\sigma\partial^\mu\sigma}{2} - \frac{m_\sigma^2}{2}\sigma^2 + \Lambda + \alpha R \\ & + \frac{1}{64\pi^2}\Big\{ \frac{1}{24}\big(R - 3g\sigma^2\big)\big(R - 3g\sigma^2 - 4m_\varphi^2\big) \\ & + \Big[-\Big(m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}\Big)^2 + \frac{G}{180}\Big] \log\Big(\frac{m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}}{m_\varphi^2}\Big)\Big\}, \end{aligned} \tag{3.18}$$

where we have used the Gauss-Bonnet density, defined as

$$G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}. \tag{3.19}$$

In the above we have only included contributions up to terms of type $\mathcal{O}(R^2/M^2)$. The Lagrangian can now be used to derive all the results we are interested in and in particular, there are two important special cases that we wish to address:

- How do the quantum corrections change the behavior of the field in the situation where there field itself is not responsible of the curvature of spacetime, but behaves only as a spectator for various choices for the scale factor $a(t)$?

- How do the quantum corrections change the dynamics of spacetime when we allow quantum back-reactions, especially for the case of inflation?

3.3.1 Spectator field dynamics in de Sitter space

For the spectator field case the assumption is that there exists some other type of matter or energy that completely dominates the energy density and, because of this, determines the

[3]The choice $g^{\mu\nu} = \eta^{\mu\nu}$ is problematic in terms of a non-zero cosmological constant, since it does not exist in Minkowski space. Even though we include Λ in the results, we assume it to be negligible.

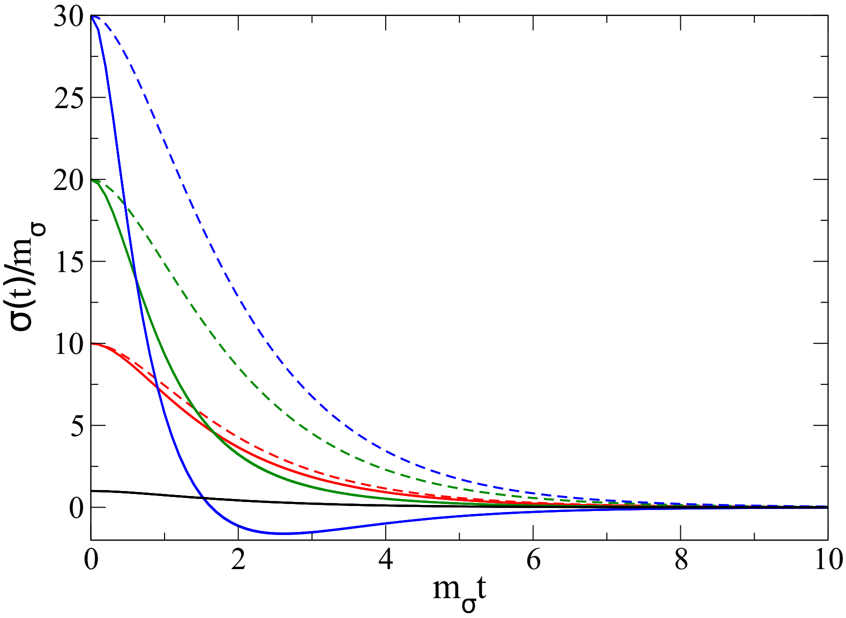

Figure 3.1: The evolution of the spectator field $\sigma(t)$ for different values of σ_0 and different approximations, in a de Sitter Universe. We use $m_\varphi/m_\sigma = 2$, $g = 1$, and $H_0/m_\sigma = \sqrt{1/2}$. The classical results for initial values of $\sigma_0/m_\sigma = 1, 10, 20, 30$ are represented by the dashed lines and the curved space quantum corrected ones by the solid lines.

evolution of the scale factor. The matter field σ merely evolves in this given background metric. This is for example how the curvaton field mentioned in section 2.2 is assumed to behave during inflation. Here we only explicitly show the de Sitter universe case, which corresponds to the scale factor

$$a(t) = e^{Ht}, \qquad H = H_0. \tag{3.20}$$

The equation of motion derived via variation is now

$$\begin{aligned}\ddot{\sigma} + 3H\dot{\sigma} + m_\sigma^2\sigma &= \frac{1}{64\pi^2}\Bigg\{\frac{g\sigma}{2}\left(2m_\varphi^2 - R + 3g\sigma^2\right) + g\sigma\frac{\left(\frac{G}{180} - \left(m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}\right)^2\right)}{m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}} \\ &\quad -2g\sigma\left(m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}\right)\log\left(\frac{m_\varphi^2 - \frac{R}{6} + \frac{g\sigma^2}{2}}{m_\varphi^2}\right)\Bigg\},\end{aligned} \tag{3.21}$$

which at least at the theoretical level has very non-trivial terms coming from the effects of curved space field theory, as is evident from the right hand side of (3.21). In order to make the analysis more comprehensive, we distinguish three levels of approximation: the classical level means simply neglecting the right hand side of (3.21), order H^0 ignores gravitational operators in the quantum corrections, order H^2 includes all occurrences of R and finally order H^4 also includes the non-Einsteinian tensors, namely G. We choose the parameters $m_\varphi/m_\sigma = 2$, $H_0/m_\sigma = \sqrt{1/2}$, $g = 1$ with the driving idea being obtaining the maximal effect possible from the quantum contributions.

In Fig. 3.1, one can find the evolution for the inflationary, de Sitter type background where the initial conditions are chosen as $\sigma_0/m_\sigma = 1, 10, 20, 30$, denoted with black, red, green and blue curves respectively; the dashed lines signify the classical tree level result and the full lines the quantum corrected ones, calculated without neglecting any gravitational contributions.

We observe that the quantum corrections are small for small initial field values, which is simply due to the overall factor $1/(64\pi^2)$, which is expected. We also find that by far the dominant contribution to the quantum dynamics comes from the Minkowski space contributions and curved space effects are insignificant. In fact, curves including back-reaction from curved space effects are indistinguishable in Fig. 3.1 from the Minkowski quantum results, i.e. the mentioned three different levels of approximation in practice make no difference. The same behavior was verified for the matter dominated and radiation dominated cases in **I**.

3.3.2 Quantum corrected dynamics for the inflaton

Now we proceed to solve the complete quantum corrected dynamics of the scale factor a for slow-roll inflation. Classically our potential in (3.21) is now of a simple quadratic form and we can write the first potential slow-roll parameter in (2.22) as

$$V(\sigma) = \frac{1}{2}m_\sigma^2\sigma^2, \qquad \epsilon_V = \frac{2M_{\rm pl}^2}{\sigma^2}. \tag{3.22}$$

Hence if we neglect quantum corrections, inflation will arise for $\sigma_0 > \sqrt{2}M_{\rm pl}$ since then we have $\epsilon_V < 1$, and we are assuming of course that condition (2.19) holds. Including the curved space quantum corrections also in the Einstein equations is now a simple task of varying the effective action formed from the Lagrangian (3.18). The quantum corrected version of the first Friedmann equation is

$$\begin{aligned}
&3\frac{\dot a^2}{a^2}\left\{1 - \frac{1}{96\pi^2 M_{\rm pl}^2}\left[\frac{g\sigma^2}{2} - \left(m_\phi^2 + \frac{g\sigma^2}{2}\right)\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right)\right]\right\}\\
&+\frac{1}{36\pi^2 M_{\rm pl}^2}\left\{\frac{\dot a}{a}\frac{\dot\sigma}{\sigma}g\sigma^2\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right)\right\}\\
&=\Lambda + \frac{1}{M_{\rm pl}^2}\left(\frac{\dot\sigma^2}{2}+\frac{m_\sigma^2}{2}\sigma^2\right)\\
&-\frac{1}{32\pi^2}\left[\frac{g\sigma^2\left(4m_\phi^2+3g\sigma^2\right)}{16} - \frac{1}{2}(m_\phi^2+\frac{g\sigma^2}{2})^2\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right)\right]
\end{aligned} \tag{3.23}$$

and the second one is

$$\begin{aligned}
&\frac{\ddot a}{a}\left\{1 + \frac{1}{192\pi^2 M_{\rm pl}^2}\left[\left(2m_\phi^2+g\sigma^2\right)\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right) - g\sigma^2\right]\right\}\\
&+\frac{1}{192\pi^2 M_{\rm pl}^2}\left\{\frac{\dot a}{a}\frac{\dot\sigma}{\sigma}g\sigma^2\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right)\right\}\\
&=-\frac{1}{3M_{\rm pl}^2}\left[\dot\sigma^2 - \frac{1}{2}m_\sigma^2\sigma^2\right] + \frac{\Lambda}{3}\\
&-\frac{1}{192\pi^2 M_{\rm pl}^2}\left\{\frac{g\sigma^2\left(4m_\phi^2+3g\sigma^2\right)}{8} - \left[\left(m_\phi^2+\frac{g\sigma^2}{2}\right)^2 - \left(g\dot\sigma^2+g\ddot\sigma\sigma\right)\right]\log\left(1+\frac{g\sigma^2}{2m_\phi^2}\right)\right\}
\end{aligned} \tag{3.24}$$

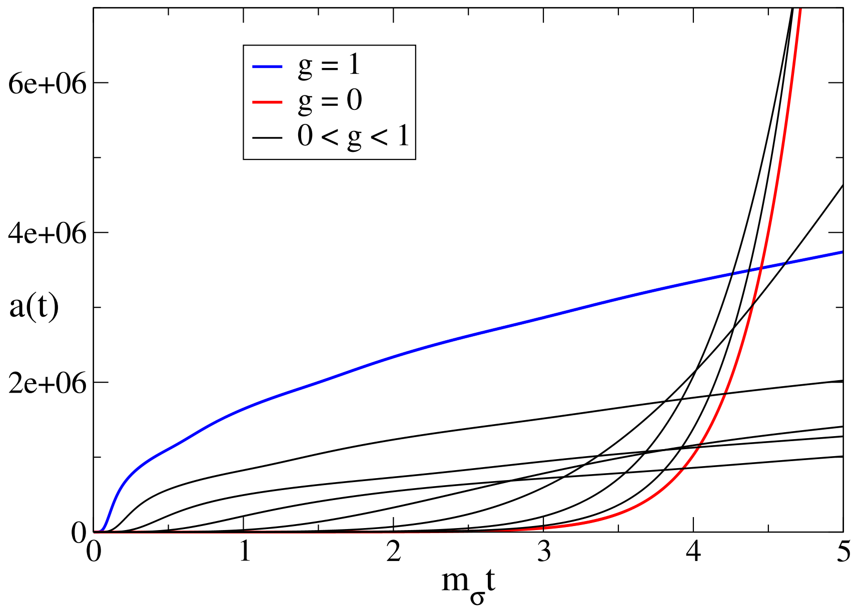

Figure 3.2: The evolution of the scale factor when $\sigma_0 = 10M_{\mathrm{pl}}$, for different values of g for $M_{\mathrm{pl}}/m_\sigma = 100$.

Neglecting the quantum contributions[4], the above equations reduce to the standard classical Friedmann equations in (2.8) and (2.9) for the potential in (3.22). The quantum corrections to the gravity equations are indeed very non-trivial. In contrast to the classical equations, we immediately see that it is no longer apparent that we can divide the equation into contributions from the matter fields and contributions from purely gravitational dynamics, as there are mixing terms of type $\dot{a}\dot{\sigma}$. An equally interesting observation is that there are now contributions from odd powers of $\dot{a}$. From this we see that for the quantum dynamics the direction of the expansion, i.e. increasing or decreasing a, may in some cases be meaningful. This is purely an effect of performing the quantum calculations in curved background, i.e. had we approximated spacetime to be flat for our quantum dynamics, none of this would be visible.

Solving the coupled equations (3.21), (3.23) and (3.24), we get the result for consistent quantum corrected dynamics. Due to the highly non-linear and coupled nature of the equations this must be done numerically. In Fig. 3.2 the evolution of $a(t)$ is presented for $\sigma_0 = 10M_{\mathrm{pl}}$ for early times. The red line represents the non-interacting $g = 0$ case where inflation is prolonged and leads to standard exponential growth of the Universe. If we then tune g towards unity, represented by the full, black lines, we see that inflation becomes increasingly weaker as we get close to $g \simeq 1$, which is denoted with blue. We can therefore deduce that quantum effects may significantly weaken inflation. This effect is again due in large part to the Minkowski space quantum corrections. However, it is open to debate, whether our renormalization point, chosen to be at zero scale, can be used for constructing a theory valid all the way up to the start of inflation.

[4]Since in our calculations $\hbar = 1$, the quantum corrections are identified by the π^{-2} prefactors.

3.4 Discussion

We can infer from the results in section (3.3) that even at the very limit of validity of the perturbative regime i.e. with a coupling constant set to unity, the curved space quantum corrections make little difference. It would appear that using the Minkowski space approximation for the loop calculations is completely adequate for all practical purposes. However, for the spectator field scenario and for the completely self-consistent solution for the matter and gravitational fields, there is a visible difference between the classical and the quantum predictions. This conclusion might still be slightly premature, since it neglects a potentially significant and a rather intricate detail concerning renormalization. Supposing one chooses to match the quantum theory to the classical theory at some scale μ, then the quantum corrections will have logarithms containing (schematically) the value of some field φ and the scale to form a dimensionless number as φ/μ. So the further one is from the renormalization point, the larger the absolute value of the logarithm appears. Since a loop expansion to a higher order will contain higher powers of logarithms, multiplied by a coupling constant, we realize that implementing our effective action for cases where $|g\log(1+g\sigma^2/(2m_\phi^2))| > 1$ is questionable. However, since inflation spans a large range of scales it is very possible that these problems surface, no matter what one chooses as the renormalization point. It may be that implementation of renormalization group improvement techniques [47] is needed. However, we have not studied the issue further in this context and it turns out that in the approach of chapter 5 RG improvement is automatically included.

Another matter we should comment is the validity range of the effective action, or more accurately the Schwinger-DeWitt expansion. To be precise, there are other methods besides the heat kernel expansion for deriving an expression for the effective action [115] (and references therein), but their mathematical complexity makes them laborious to use in practice (see [116] for an example). In deriving the result (3.13), we used the expansion (3.11) which is an expansion around small proper time τ. It can be seen by inserting (3.9) into (3.8) that the divergences in the integral occur at $\tau = 0$ when $n = 4$ and so the region near $\tau = 0$ corresponds to the ultraviolet region of the theory. We can thus conclude that a small proper time expansion is only correct in terms of the ultraviolet behavior of the theory and the infrared contributions corresponding to large τ are exponentially damped as can be seen from the ansatz (3.9). Conversely as already stated, the terms in (3.13) are formed from an increasing number derivatives of the matter and gravitational fields, which is also evident by dimensional reasons from the increasing inverse power of the effective mass (3.13). So a more descriptive way of expression the validity of this expansion is that on the scale of the effective mass the fields must be *small and slowly varying*. If our physics is dominated by the ultraviolet dynamics - which is sometimes assumed - then the heat kernel approach will be a trustworthy approximation for the effective equations of motion. In a situation where the contribution of the infrared region plays a significant role we must find another method to suit our purposes. But if one wishes to be certain of the dominance of the ultaviolet regime, and hence the validity of the Schwinger-DeWitt expansion, we must know the size of the infrared contribution. The most desirable method for determining this would of course be to actually perform the calculation of the infrared portion for the quantum corrections. This is done in **III** (see references for previous works), but it first required some tools that were developed in **II**.

The key observation for this was realizing that if we could consistently perform the entire calculation at the equation of motion level, there we can acquire a significant simplification by setting $g^{\mu\nu}$ to be of the FRW form.

Chapter 4

Renormalization of the equations of motion in curved spacetime

Since in we are interested in calculating corrections to inflationary physics, we can restrict ourselves to a homogeneous and isotropic space. This immediately gives the idea that we might hope to find a significant simplification if we can find a way to constrain our metric to be of the FRW form throughout the calculation. Unfortunately the effective action $\Gamma^{(1)}[\varphi, g^{\mu\nu}]$ is defined with respect to a general metric. This stems from the fact that we must vary with respect to a general metric, as in (2.42), in order to get to the Einstein equation. Imposing constraints – or boundary conditions – on the metric at the level of the action gives additional complications. This implies that, instead of working with the effective action, one should perform the entire analysis for the equations of motion, where the restriction to FRW type metric is perfectly allowed. So instead of first calculating the effective action one may vary the quantized action $S[\hat{\varphi}, g^{\mu\nu}]$ with respect to the operators as (2.43) and only afterwards calculate the expectation values.

This leads to a problem. If we wish to perform the entire calculation at the equation of motion level, then there are not many renormalization techniques available for the curved equations of motion, especially when interactions are included. Renormalization is such an important process in interacting theories, giving rise to intricate phenomena such as the running of the couplings, that in order to derive robust predictions for the quantum corrections we insist on being able to perform it consistently in curved spacetime.

The early work on renormalization in curved backgrounds in most cases concentrated on the consistent cancellation of divergences, without explicitly calculating the finite remainder of the counterterms [113, 117, 118, 119, 120, 121] (however, see also [122, 123, 124]). These approaches are of essential theoretical value, but they do not provide us with a procedure with which to calculate the results with correct finite parts. Since our main focus is in studying inflationary physics, we are particularly interested in a method that allows us to derive the correct finite parts of the counter terms for a spacetime with a de Sitter type of behavior, which is quite distinct from an expansion around Minkowski space. Also, in order to obtain quantitative physical predictions, the renormalization scale must be known and one must be able to fix it freely in order to assign a proper physical interpretation for the constants of the theory at the scales being studied.

In the past, if renormalizing at the level of the action was not a viable option then the only practical method available for renormalizing at the level of the equations of motion was adiabatic subtraction [125, 126, 127, 128] (see also [129]), having recently been applied in [130, 131, 132]. However, adiabatic subtraction has been primarily used for non-interacting theories and is, strictly speaking, a regularization method. When implemented for interacting theories it has limitations. Such as, a lack of an explicit renormalization scale and the fact that for an interacting theory the counter terms cannot be reduced to a redefinition of the constants in the classical action.

Finding a method for renormalization at the level of the equations of motion, with finite parts suitable to a de Sitter space and adjustable renromalization scale for each constant led us eventually to the procedure explained in **II**. In addition to introducing the renormalization method in **II** we also used it in practice, for calculating the fourth adiabatic order equations of motion for the standard φ^4 scalar field theory with the potential (2.38) in the adiabatic vacuum. The adiabatic vacuum is an expansion in terms of derivatives, so it is closely related to the Schwinger-DeWitt expansion of section 3.1 for a metric of the FRW form. The reason for choosing the adiabatic vacuum was to show that it is generally simpler to perform the calculation at the equation of motion level and that this approach gives equivalent results to the Schwinger-DeWitt expansion. Results for the physical quantities in the adiabatic vacuum also provide a consistency check of the validity of adiabatic subtraction for interacting theories. We start by discussing the adiabatic vacuum.

4.1 Adiabatic vacuum

We must first of course define what we mean by an adiabatic vacuum. Again, as in chapter 2, we define the fluctuation of the operator $\hat{\varphi}$ as $\hat{\varphi} \to \varphi + \hat{\phi}$ with $\langle\hat{\varphi}\rangle \equiv \varphi$. We can start with a field equation of motion, similar to (2.33) of the form

$$\Big[-\Box + M^2\Big]\hat{\phi} = 0, \tag{4.1}$$

where M is some arbitrary and possibly time- and φ-dependent mass parameter, in our case the effective mass in (2.32). We can solve this in terms of mode functions by using an ansatz

$$\hat{\phi} = \int d^{n-1}k\big[a_{\mathbf{k}}u_{\mathbf{k}} + a^*_{\mathbf{k}}u^*_{\mathbf{k}}\big], \quad u_{\mathbf{k}} = \frac{1}{\sqrt{2(2\pi)^{n-1}a^{n-1}}}h_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$h_{\mathbf{k}}(t) = \frac{1}{\sqrt{W}}e^{-i\int^t W dt'}, \tag{4.2}$$

with the standard commutation relations (2.35). Assuming that our metric is of the FRW form (2.1), we can write W as an adiabatic expansion, i.e an expansion in "dots"

$$W = c_0 + c_1\frac{\dot{a}}{a} + c_2\frac{\dot{M}}{M} + c_3\frac{\dot{a}^2}{a^2} + c_4\frac{\dot{M}^2}{M^2} + c_5\frac{\ddot{a}}{a} + c_6\frac{\ddot{M}}{M} + c_7\frac{\dot{a}\dot{M}}{aM} + \cdots, \tag{4.3}$$

with c_i being functions of M and a. As the above clearly shows, this expansion is meaningful only when φ and a are slowly varying. As can be seen from the appendixes of **II**, the coefficients c_i have increasing inverse powers of the momentum variable $\mathbf{k}$, so the approximation works better for the high momentum modes. Hence this procedure gives correct results at the ultraviolet limit and thus has very similar behavior (and limitations) to the Schwinger-DeWitt technique introduced in section (3.1).

The solution for $u_{\mathbf{k}}$ will naturally also be an expansion in the number of time derivatives and the Ath order approximate solution will include all terms with an A number of derivatives[1] and it will be denoted as $u_{\mathbf{k}}^{(A)}$. Similarly, the vacuum it defines is written as $|0^{(A)}\rangle$ and any correlator calculated in this state is called the Ath order approximation of this quantity. These approximate modes can be used to define an exact solution to the equation of motion (4.1) through the relation

$$u_{\mathbf{k}} = \alpha_{\mathbf{k}}(t)u_{\mathbf{k}}^{(A)} + \beta_{\mathbf{k}}(t)u_{\mathbf{k}}^{(A)*}, \tag{4.4}$$

where $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ must be constant in t to order A, but they may have dependence in $\mathbf{k}$. We can then choose to fix the exact mode $u_{\mathbf{k}}$ at some point $t = t_0$ to be the Ath order positive solution:

$$\alpha_{\mathbf{k}}(t_0) = 1 + \mathcal{O}(A+1) \quad \beta_{\mathbf{k}}(t_0) = 0 + \mathcal{O}(A+1) \tag{4.5}$$

This defines the Ath order adiabatic vacuum.

4.2 Adiabatic subtraction

Adiabatic subtraction [125, 126, 127, 128] is one of the most used renormalization methods for curved spacetime calculations, which is due in large part to its practicality. Strictly speaking, adiabatic subtraction is a method with which one may render the quantum contributions finite, i.e. a regularzation method, but it is often also used as a complete renormalization prescription. It has found the most use in renormalization of the energy-momentum tensor. The renormalized quantities are defined at the level of the equations of motion by subtracting an Ath order derivative approximation, explained in section 4.1, of the quantity of interest from the divergent quantum expression.

The order of the expansion A depends on the adiabatic order of divergences in the bare quantity. Thus in this procedure one only calculates a single subtraction term that includes all the divergences appearing in a particular expression thus making it finite, and counterterms such as δm^2 or $\delta\lambda$ never explicitly enter the picture. As an example we use the adiabatic subtraction procedure to renormalize the variance of a field and the quantum part of the energy-momentum tensor. The renormalized expressions defined via adiabatic subtraction read

$$\langle\hat{\varphi}^2\rangle = \langle\hat{\varphi}^2\rangle_0 + \delta\varphi^2 \equiv \langle\hat{\varphi}^2\rangle_0 - \langle 0^{(A)}|\hat{\varphi}^2|0^{(A)}\rangle\big|_{A=2} \tag{4.6}$$

$$\langle\hat{T}^Q_{\mu\nu}\rangle = \langle\hat{T}^Q_{\mu\nu}\rangle_0 + \delta T_{\mu\nu} \equiv \langle\hat{T}^Q_{\mu\nu}\rangle_0 - \langle 0^{(A)}|\hat{T}^Q_{\mu\nu}|0^{(A)}\rangle\big|_{A=4}. \tag{4.7}$$

The argument is that all the constants are taken to be renormalized since the divergences are taken care of by the adiabatic subtraction terms.

Because this procedure operates at the equation of motion level, it has all the advantages discussed at the beginning of this chapter. Also, since the counter term is an energy-momentum tensor calculated in some particular state, it poses no problems for covariant conservation of energy-momentum, which is often not the case when a subtraction term is introduced by hand.

This method has some drawbacks, however. Because the renormalized expression is derived by a single subtraction, the renormalization conditions for each constant including the renormalization scale, are not explicit. This was not a problem when the renormalization conditions were defined at the level of the action (3.14). Not clearly stating the

[1]This means that $\dot{a}^2$ and $\ddot{a}$ are of the same adiabatic order, for example.

renormalization scales leaves the finite part of each counterterm implicit and thus the physical interpretations for the renormalized constants become more difficult, especially when interactions are included. Furthermore, it is not immediately obvious that this one big subtraction term defined via an expectation value corresponds to a consistent renormalization scheme, where in the one-loop approximation all the counter terms are terms of the classical action, with possibly divergent coefficients.

In fact, as was shown in **II**, for a non-constant mean field the subtraction term defined by (4.7) can be reduced to a redefinition of the classical action *only* when the theory is non-interacting. In the alternative method introduced in **II** to be discussed in the next section, these issues are not present.

4.3 Consistent renormalization via the energy-momentum tensor

In this section we present a renormalization procedure that gives explicit control over the finite parts of the counterterms while working at the level of the equations of motion. In order to make the procedure more concrete, we will perform our calculation for the theory defined in (2.48), i.e. for a theory with the action

$$S_m[\varphi, g^{\mu\nu}] = -\int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\xi R\varphi^2 + \frac{\lambda}{4!}\varphi^4\right]. \tag{4.8}$$

We would now like to find a generalization of the equations (3.14) for the energy-momentum tensor, so that we can determine each counterterm in the action separately. For this we can use equations very similar to those in (3.14). The only difference being that we use matching between the expectation value of the energy-momentum tensor - instead of the effective action - to the classical one for determining the counterterms. In the Einstein equation defined in (2.51), with the parametrization in (2.53),

$$\langle\hat{T}_{\mu\nu}\rangle \equiv T^C_{\mu\nu} + \langle\hat{T}^Q_{\mu\nu}\rangle + \delta T^m_{\mu\nu} - \delta T^g_{\mu\nu}, \tag{4.9}$$

we now have the following results for the classical contribution $T^C_{\mu\nu}$

$$\begin{aligned} T^C_{\mu\nu} = &-\frac{g_{\mu\nu}}{2}\left[\partial_\rho\varphi\partial^\rho\varphi + m^2\varphi^2 + 2\frac{\lambda}{4!}\varphi^4\right] + \partial_\mu\varphi\partial_\nu\varphi \\ &+ \xi\big[G_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\big]\varphi^2 \end{aligned} \tag{4.10}$$

and the quantum contribution $\langle\hat{T}^Q_{\mu\nu}\rangle$

$$\begin{aligned} \langle\hat{T}^Q_{\mu\nu}\rangle = &-\frac{g_{\mu\nu}}{2}\left[\frac{\partial}{\partial x_\rho}\frac{\partial}{\partial y^\rho} + M^2\right]G(x,y)\big|_{x=y} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}G(x,y)\big|_{x=y} \\ &+ \xi\big[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\big]G(x,x), \end{aligned} \tag{4.11}$$

with the propagator defined in (2.36). The matter counter term part for the theory (4.8) is

$$\begin{aligned} \delta T^m_{\mu\nu} = &-\frac{g_{\mu\nu}}{2}\left[\delta Z\partial_\rho\varphi\partial^\rho\varphi + \delta m^2\varphi^2 + 2\frac{\delta\lambda}{4!}\varphi^4\right] + \delta\xi\big[G_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\big]\varphi^2 \\ &+ \delta Z\partial_\mu\varphi\partial_\nu\varphi, \end{aligned} \tag{4.12}$$

and the gravitational part of the counter term $\delta T^g_{\mu\nu}$ comes from (2.52)

$$\delta T^g_{\mu\nu} = -g_{\mu\nu}\delta\Lambda + 2\delta\alpha G_{\mu\nu} + 2\delta\beta \,{}^{(1)}H_{\mu\nu} + 2\delta\epsilon_1 \,{}^{(2)}H_{\mu\nu} + 2\delta\epsilon_2 H_{\mu\nu} \tag{4.13}$$

As already discussed, when working at the level of the equations of motion we can constrain the metric. Therefore we can choose that we are in a homogeneous and isotropic space with a FRW type metric (2.1), and use this accordingly in the renormalization conditions.

Now we are ready to write the renormalization equations for the coupling constants in (2.46) and (2.47). Since in section 4.4 we will calculate the energy-momentum tensor in the adiabatic vacuum, which we then wish to compare to the Schwinger-DeWitt results of chapter 3, we will here use the same renormalization scale choices of Minkowski space with a vanishing field,

$$\varphi \equiv 0, \qquad a \equiv 1. \tag{4.14}$$

From (4.10) we get the classical energy-density with the help of the tensor formulae from appendix A

$$T^C_{00} = \frac{1}{2}\left[\dot\varphi^2 + m^2\varphi^2 + 2\frac{\lambda}{4!}\varphi^4\right] + \xi\left[(n-1)\left(\frac{n}{2}-1\right)\left(\frac{\dot a}{a}\right)^2\varphi^2 + 2(n-1)\frac{\dot a}{a}\dot\varphi\varphi\right]. \tag{4.15}$$

We have written the expression in n-dimensions, so that the equations would also be valid for arbitrary dimensions, which is important if dimensional regularization is used. Analogously to (3.14) we can thus write the renormalization conditions for the "00" component of the full energy-momentum tensor by matching it to the classical value using

$$\begin{aligned}
&\left.\frac{\partial^2\langle\hat T_{00}\rangle}{\partial\dot\varphi^2}\right|_{\psi_i=\mu_i} = 1, &&\left.\frac{\partial^2\langle\hat T_{00}\rangle}{\partial\varphi^2}\right|_{\psi_i=\mu_i} = m^2, &&\left.\frac{\partial^4\langle\hat T_{00}\rangle}{\partial\varphi^4}\right|_{\psi_i=\mu_i} = \lambda,\\
&\left.\frac{\partial^3\langle\hat T_{00}\rangle}{\partial\varphi\partial\dot\varphi\partial(\dot a/a)}\right|_{\psi_i=\mu_i} = 2\xi(n-1) &&\langle\hat T_{00}\rangle\big|_{\psi_i=\mu_i} = 0, &&\left.\frac{\partial^2\langle\hat T_{00}\rangle}{\partial(\dot a/a)^2}\right|_{\psi_i=\mu_i} = 0\\
&\left.\frac{\partial^3\langle\hat T_{00}\rangle}{\partial(\dot a/a)^2\partial(\ddot a/a)}\right|_{\psi_i=\mu_i} = 0, &&\left.\frac{\partial^2\langle\hat T_{00}\rangle}{\partial(\ddot a/a)^2}\right|_{\psi_i=\mu_i} = 0, &&\left.\frac{\partial^4\langle\hat T_{00}\rangle}{\partial(\dot a/a)^4}\right|_{\psi_i=\mu_i} = 0.
\end{aligned} \tag{4.16}$$

The first four conditions in the above fix the counterterms coming from the matter part of the action and the rest do the same for the counterterms from the gravity part. There are of course other possible choices for renormalization equations. For example $\delta\xi$ can also be determined from

$$\left.\frac{\partial^4\langle\hat T_{00}\rangle}{\partial\varphi^2\partial(\dot a/a)^2}\right|_{\psi_i=\mu_i} = 4\xi(n-1)\left(\frac{n}{2}-1\right). \tag{4.17}$$

In a similar fashion, we could have chosen other components of $T_{\mu\nu}$ for determining the renormalization constants. In complete generality, the right hand side of the conditions could have been written with the classical energy-momentum evaluated at the chosen scales, paralleling the conditions in (3.14), but here this is not relevant for the scale choices in (4.14).

A few comments are now in order. Here we have renormalized by simply including counterterms in the constants of the original action, which means that covariant conservation is automatically satisfied[2]. Also, by *assuming* the equations (4.16) to work we

[2]This can be shown by operating with ∇^μ on the right hand side side of (2.51) and using the Bianchi identities and commutator formulae.

implicitly assumed that all the equations are analytic at the chosen scales μ_i. Famously in [50] it was shown that in the φ^4 theory there is an infrared singularity in the massless limit. To bypass this issue at the renormalization stage, one may choose non-zero renormalization scales μ_i. However, this is a non-trivial issue, since changing renormalization scales changes the definitions of the constants and ultimately alters the range in parameter space where the perturbative expansion can be trusted.

4.4 Deriving the second order adiabatic energy-density

As an example, we now derive the energy-density in the adiabatic vacuum and renormalize it by using the technique discussed in section 4.3. Using the expressions for the mode in (2.34) and the commutator formulae in (2.35), we can write the quantum energy-density from (4.11) as

$$\langle \hat{T}^{Q}_{00} \rangle = \int d^{n-1}k \bigg\{ \frac{1}{2}\bigg[|\dot{u}_{\mathbf{k}}|^2 + \Big(k^2/a^2 + M_0^2\Big)|u_{\mathbf{k}}|^2 \bigg] + \xi \bigg[G_{00} + (n-1)\frac{\dot{a}}{a}\partial_0 \bigg] |u_{\mathbf{k}}|^2 \bigg\} \quad (4.18)$$

where we have defined

$$M_0^2 \equiv m^2 + \frac{\lambda}{2}\varphi^2. \quad (4.19)$$

By inserting in the above expression the adiabatic mode discussed in section 4.1, where the explicit (and quite complicated) result for the adiabatic phase is found from the appendixes of **II**, we get the result for "00" components of the quantum part of the energy-momentum tensor. This result can then be renormalized with the equations (4.16). After deriving the counter terms, which may also explicitly be found from **II**, we can write the finite energy momentum tensor to second adiabatic order

$$\begin{aligned} \underline{\langle \hat{T}^{Q}_{00} \rangle} = \frac{1}{64\pi^2} \bigg\{ & (1-6\xi)\lambda\varphi^2 \frac{\dot{a}^2}{a^2} - \frac{3}{8}\lambda^2\varphi^4 - \frac{\lambda^2 m^2 \varphi^4}{4M_0^2} + \frac{\lambda\varphi^2\big(-3m^4 + \lambda\dot{\varphi}^2\big)}{6M_0^2} \\ & + \log\left(\frac{M_0^2}{m^2}\right) \bigg[M_0^4 - 2(1-6\xi)M_0^2\frac{\dot{a}^2}{a^2} - (1-6\xi)2\lambda\frac{\varphi\dot{\varphi}\dot{a}}{a} \bigg] \bigg\} + \mathcal{O}(A)^4. \end{aligned} \quad (4.20)$$

If we set $\xi \to 0$, $m \to m_\phi$, $\varphi \to \sigma$ and $\lambda \to g$ this result coincides with the quantum contribution of the energy-density calculated with the Schwinger-DeWitt expansion in equation (3.23)[3]. Similar results also apply for the pressure density and the field equation of motion.

4.5 Discussion

In **II** we used the above technique to derive the equations of motion in the fourth order adiabatic vacuum. There we also derived the conformal anomaly, without any reference to an effective action. This was simply for checking that renormalization is implemented correctly and we also wanted to emphasize the simplification that arises when one may constrain the metric to be of the FRW form instead of a general $g^{\mu\nu}$. One may get an idea of how complicated the Schwinger-DeWitt coefficients become after the first few orders from [133]. The idea put forth was not to advocate the use of the adiabatic vacuum for the calculation of counter terms in an arbitrary metric. Rather, we merely used the adiabatic

[3]There is an extra term in (4.20), which is included here, since it is second order in the adiabatic expansion but third order in the Schwinger-DeWitt expansion.

vacuum as an example because it gave us a direct way of comparing and checking our method against well known results since after all, the adiabatic vacuum is an expansion in gradients just like the Schwinger-DeWitt expansion. The most important element here is the fact that for the finite parts of the renormalization constants one uses the same background spacetime for calculating the counter terms, which is used for the respective problem. For example, renormalizing the cosmological constant to a finite value by using the effective action calculated via the Schwinger-DeWittt expansion as in section 3 is problematic since the Schwinger-DeWitt expansion is an expansion around Minkowski space and in such a space a large cosmological constant does not exist. However, by using renormalization at the equation of motion level we can simply calculate the energy-momentum tensor with a spactime ansatz suited for a non-zero cosmological constant and use that in the conditions (4.16) for obtaining the counter term. This kind of an approach is challenging when working at the level of the action. Of course, the hope is that besides inflation our technique could be put to use in other problems of curved space field theory, such as the cosmological constant (CC) problem [134]. At the moment this is little more than pure speculation and whether or not our technique provides new insights for the CC problem requires detailed calculations. We will return to this issue in the concluding section of the thesis.

The technique described in this chapter was put to use in **III** where the entire calculation was performed without any reference to an effective action.

Chapter 5

Effective equations of motion in the slow-roll approximation

In the previous section, we derived a renomalization procedure that allows us to perform consistent renormalization completely at the equation of motion level. Next we seek to find equations of motion for inflation that would incorporate behavior not seen by the Schwinger-DeWitt expansion. This means that we hope to gain some information of the infrared dynamics. Because our main interest is to study quantum effects in inflation, the natural choice is to use a slow-roll type expansion from section 2.3 for our calculation. The core difference to our previous calculation in chapter 3 is that we are now using an expansion around de Sitter space. This is somewhat more challenging than using the Schwinger-DeWitt or adiabatic approaches.

Several works addressing similar issues and with similar approaches already existed before **III**. In particular, in **III** we generalized the results of [135, 136, 137, 138]. Previously, a de Sitter calculation was done in [135, 136, 137, 139, 140, 141, 142, 143, 144, 145, 146, 147] and where in [136, 140, 141, 142] nonperturbative summation techniques, to be explained in section 5.4, were used (see also [148, 149]). In [138, 150], the 1PI approximation was used to first order in slow-roll.

Initially we were only pursuing the one-loop corrections to first order in the slow-roll parameters. However, the result showed infrared divergent behavior and forced us to improve our loop calculation. Here we had merely discovered a version of the already well-known infrared divergence in de Sitter space, which is reviewed for cosmological correlators in [100]. An infrared divergence is something one frequently encounters in finite temperature field theory and is usually a sign that one must *resum* the loop expansion. In the cosmological context it was first shown in [151, 152] that resummation cures an infrared divergence for a scalar field theory with a de Sitter background and a quadratic interaction term. In practice one implements this by instead of using a free propagator for the perturbation theory, one includes contributions from interactions in the propagator, in other words "dresses" it. Hopefully, this is enough to tame the infrared poles. There are a number of schemes with which to do the re-summation and we chose to use the 2PI technique to first non-trivial order. This truncation is commonly known as the Hartree approximation.

Due to the highly coupled nature of the equations of motion, we were only able to derive the leading infrared contribution in addition to the already known ultraviolet terms.

5.1 Vacuum to first order in slow-roll

Now we proceed to write the equation for the quantum mode in (2.33) for the standard φ^4 theory defined in (4.8) as an expansion in the slow-roll parameters of section (2.3). As our expansion parameters we will be using the first Hubble slow-roll parameter from (2.17)

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \tag{5.1}$$

the second Hubble slow-roll parameter from (2.21)

$$\delta_H \equiv \frac{\ddot{H}}{2H\dot{H}} \tag{5.2}$$

and a parameter closely related to the second potential Hubble slow-roll parameter in (2.22)[1]

$$\delta \equiv \frac{M^2}{H^2}, \tag{5.3}$$

where M is the effective mass of (2.32). Our aim is an accuracy up to first order in ϵ and δ, and the leading infrared contribution. However, as we shall see the infrared momentum region gives contributions proportional to the inverse of δ and ϵ, as already noticed in [138]. Hence we will include higher order terms in our quantum modes in section 5.4 in order to achieve the desired accuracy.

From the definition (5.1) one can easily see that for the derivative of ϵ we have the relation

$$\dot{\epsilon} = 2\epsilon\big(\epsilon + \delta_H\big), \tag{5.4}$$

so it is higher order in the expansion and expressible with the first order slow-roll parameters. If we then further make the definitions

$$x \equiv \frac{|\mathbf{k}|}{aH(1-\epsilon)}, \qquad u_{\mathbf{k}} = \frac{1}{\sqrt{2(2\pi)^{n-1}a^{n-1}}} h_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad h_{\mathbf{k}} \equiv \sqrt{\frac{\pi}{2H(1-\epsilon)}}\bar{h}_{\mathbf{k}},$$
$$\nu^2 \equiv \frac{(n-3\epsilon-1)(n+\epsilon-1)}{4(1-\epsilon)^2} - \frac{\delta_H\epsilon+\delta}{(1-\epsilon)^2}, \tag{5.5}$$

we can write the mode equation (2.33) to quadratic order in the slow-roll parameters as

$$x^2\frac{d^2\bar{h}_{\mathbf{k}}(t)}{dx^2} + x\frac{d\bar{h}_{\mathbf{k}}(t)}{dx} + \big(x^2-\nu^2\big)\bar{h}_{\mathbf{k}}(t) = 0. \tag{5.6}$$

If we take the limit $\epsilon \to 0$ and a constant ν this equation is the standard Bessel equation[2], whose solution can be written as a linear combination of the Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$. As a boundary condition to fix our mode solutions, we impose that the mode corresponds to the positive frequency mode at high momentum to first order in ϵ namely

$$h_{\mathbf{k}}(t) \to \frac{e^{-i\int^t \omega(t')dt'}}{\sqrt{\omega(t)}}, \qquad \omega(t) \to \frac{k}{a} \tag{5.7}$$

[1] By using the zeroth order slow-roll version of the first Friedmann one can see that this choice is proportional to (2.23). i.e. $\delta \equiv 3M_{\rm pl}^2\frac{V''(\varphi)}{V(\varphi)}$.

[2] This equation is often written in terms of conformal time $dt = ad\eta$, which gives $f''_{\mathbf{k}}(\eta) + \big[k^2 + (\nu^2 - 1/4)/\eta^2\big]f_{\mathbf{k}}(\eta) = 0$, for $u_{\mathbf{k}} = a^{\frac{n-2}{2}} f_{\mathbf{k}}(\eta)$.

at $k \to \infty$. Asymptotic behaviour of the Hankel function for large argument then allows us to write the solution

$$h_{\mathbf{k}}(t) = \sqrt{\frac{\pi}{2H(1-\epsilon)}}\Big[C_1(k)H_\nu^{(1)}(x) + C_2(k)H_\nu^{(2)}(x)\Big]. \tag{5.8}$$

Completely fixing $C_{1,2}$ requires an additional boundary condition and we make the standard choice [36] of setting $C_2 \to 0$ and $C_1 \to 1$, which corresponds to the Bunch-Davies vacuum solution [153]. There also exist studies where the effects of different boundary conditions is analyzed [154] (and references therein), but we will be content with the Bunch-Davies solution.

5.2 Improvement of the previous result: leading infrared term

Let us now proceed to quantify the approximation used in this calculation. As we already discussed at length on several occasions, the Schwinger-DeWitt expansion does not see all infrared contributions correctly. By using the slow-roll approximation for the mode from section 5.1 we hope to gain some additional insight of the infrared behaviour. Our aim is an accuracy up to linear order in the slow-roll parameters for the ultraviolet and the leading infrared terms.

As can be shown by explicitly evaluating the infrared integrals, evaluated in **III**, one obtains terms proportional to

$$\frac{\delta}{3-2\nu}, \qquad \frac{\epsilon}{3-2\nu} \tag{5.9}$$

with ν defined in (5.5). In order to facilitate the analytic use of our results for the above expression we have approximated it by a series expansion in the slow-roll parameters

$$\frac{1}{3-2\nu} \approx \frac{3}{2\big(\delta - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon\big)} + \cdots . \tag{5.10}$$

In our calculation we will also encounter derivatives of the contribution $(3-2\nu)^{-1}$ giving us terms such as

$$\partial_t \frac{1}{3-2\nu}, \qquad (\partial_t)^2\frac{1}{3-2\nu}, \tag{5.11}$$

and hence the terms in (5.9) are leading only if we have sufficiently small derivatives for δ, δ_H and ϵ, which we assume to be the case in our analysis. So to summarize, the terms in (5.9) are considered leading and are included, and terms such as (5.11) are considered sub-leading and are thus neglected. We also neglect leading terms multiplied with powers of δ and/or ϵ.

As for the terms that are expressible as a power series in the slow-roll parameters, i.e. not coming from the infrared, we simply include them up to linear orders in δ and ϵ in the quantum corrections. However, for the divergent pieces as a check of consistent renormalization we have included the δ^2 and $\epsilon\delta$ contributions. Importantly, we make no approximations for contributions from the classical part. In section 5.4 we use the same approximations, but for the quantity $\delta_{2\mathrm{PI}}$, which is defined as in (5.3), but with the re-summed effective mass in the numerator, to be discussed in section (5.4.1).

5.3 1PI effective equations of motion

In this section we will derive the quantum corrected finite equations of motion in the vacuum defined by the mode in (5.8). We have written the unrenormalized one-loop equations for the φ^4 theory already in sections 2.4 and 4.3 but for completeness, we write them here once more. Note that in this section all the counter terms are defined to include only the $(n-4)^{-1}$ type poles and thus in order to derive the equations with physical constants one must include a set of *finite* counter terms, which will be done in section 5.5.

The field equation is the result of a variation of the quantized action in (4.8) with respect to the field φ and reads

$$\begin{aligned} &\Big[-\Box+m^2+\delta m^2+(\xi+\delta\xi)R\Big]\varphi+\frac{\lambda+\delta\lambda}{3!}\varphi^3+\frac{\lambda}{2}\varphi\langle\hat{\phi}^2\rangle=0 \\ \Leftrightarrow\qquad &\Big[-\Box+m^2+\xi R\Big]\varphi+\frac{\lambda}{3!}\varphi^3+\frac{\lambda}{2}\varphi\langle\underline{\hat{\phi}^2}\rangle=0, \end{aligned} \tag{5.12}$$

whereas in (2.53) the underline signifies a quantity which includes the quantum corrections and the infinite parts of the counter terms. Similarly we can write from the variation of the action

$$\begin{aligned} \frac{1}{8\pi G}(\Lambda g_{\mu\nu}+G_{\mu\nu})=T_{\mu\nu}&\equiv T^C_{\mu\nu}+\langle\hat{T}^Q_{\mu\nu}\rangle+\delta T_{\mu\nu} \\ &\equiv T^C_{\mu\nu}+\langle\underline{\hat{T}^Q_{\mu\nu}}\rangle, \end{aligned} \tag{5.13}$$

where the classical and quantum pieces are given in (4.10) and (4.11). The energy-momentum counter term is divided into two parts, as already shown in section 2.5 as

$$\delta T_{\mu\nu}\equiv\delta T^m_{\mu\nu}-T^g_{\mu\nu} \tag{5.14}$$

with the matter piece written in (4.12) and the gravity piece in (4.13).

The next issue is how to obtain expressions for the loop $\langle\hat{\phi}^2\rangle=G(x,x)$ and the quantum energy-momentum $\langle\hat{T}^Q_{\mu\nu}\rangle$ in equations (5.12) and (5.13). The calculation here follows closely the steps outlined in [136] and here we only sketch the derivation, where the details can be found in **III**. We will essentially use a slow-roll expansion in the parameters discussed in section 5.1. For example, from the definitions of section 5.1 we can write an expression for the loop via the first Hankel function

$$\langle\hat{\phi}^2\rangle=\int d^{n-1}|k|\,|u_{\mathbf{k}}|^2=\frac{\mu^{4-n}\sqrt{\pi}}{4\Gamma[\frac{n-1}{2}]}\bigg(\frac{(1-\epsilon)H}{2\sqrt{\pi}}\bigg)^{n-2}\int_0^\infty dx\;x^{n-2}|H^{(1)}_\nu(x)|^2. \tag{5.15}$$

We then split the integration into three regions

$$x<\kappa_{IR},\quad \kappa_{IR}<x<\kappa_{UV},\quad \kappa_{UV}<x, \tag{5.16}$$

with the parameters

$$\kappa_{IR}\ll 1\ll\kappa_{UV}. \tag{5.17}$$

For the infrared region defined we use a small momentum asymptotic expansion of the Hankel function and, analogously, for the ultraviolet contribution we use a high momentum asymptotic expansion. As for the intermediate region between κ_{IR} and κ_{UV} we simply set

$\nu \to 3/2$ making an error of $\mathcal{O}(\epsilon, \delta)$. For the ultraviolet contribution, which is divergent, we use dimensional regularization instead of a cut-off, in contrary to [136]. This is because a cut-off introduces divergences that cannot be removed by covariant counter terms [155] (and references therein). Our momentum splitting procedure also has the desirable feature that the infrared region is identical to what one would obtain by using a cut-off. Effects of a cut-off in curved space are studied in more detail in [156].

Performing the calculation, we find the result for the loop $\langle\hat{\phi}^2\rangle$

$$\langle\hat{\phi}^2\rangle = \frac{H^2}{8\pi^2}\left\{(-\delta-\epsilon+2)\left[\frac{1}{4-n} - \log\left(\frac{H}{\mu}\right)\right] + \frac{3}{\delta - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}\right\}, \tag{5.18}$$

where where μ is an arbitrary renormalization scale and according to our approximation of section 5.2 we have included the leading infrared terms and neglected the linear orders in ϵ and δ, except when appearing with the divergence.

Similarly we can write the result for the quantum energy-momentum

$$\begin{aligned}\langle\hat{T}^Q_{\mu\nu}\rangle = -g_{\mu\nu}\frac{H^4}{32\pi^2}&\left\{\left(-\delta^2 - 4\delta\epsilon + 2\delta + 6\epsilon\right)\left[\frac{1}{4-n} - \log\left(\frac{H}{\mu}\right)\right] + \frac{6\delta}{\delta - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}\right\}\\ &+ \xi\left[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\right]\langle\hat{\phi}^2\rangle,\end{aligned} \tag{5.19}$$

In the above expression the accuracy is to leading order in the slow-roll parameters for the ultraviolet contributions and the leading infrared term. The infrared effects come from the $(\delta - 3\epsilon + \mathcal{O}(\epsilon^2))^{-1}$-type terms in (5.18 – 5.19). The reason we have chosen not to include any term of type $\propto H^2\delta$ or $H^2\epsilon$ is that they can always be completely absorbed in the counter terms and hence are physically irrelevant. For a proof of this statement, see the appendixes of **III**.

In order to remove the divergent $(4-n)^{-1}$ poles from the results (5.19) and (5.18), we can again use the equations in (4.16). The scale choices are now irrelevant since we are only interested in the divergent parts. This calculation is an easy exercise in linear algebra and gives the result that indeed all the poles are cancelled by the counter terms coming from the classical action. Hence we can write the divergence-free result for the field equation of motion (5.12)

$$\begin{aligned}&\ddot{\varphi} + 3H\dot{\varphi} + \xi R\varphi + m^2\varphi + \frac{\lambda}{6}\varphi^3\\ &\quad + \frac{\lambda\varphi H^2}{16\pi^2}\left\{(\delta+\epsilon-2)\log\left(\frac{H}{\mu}\right) + \frac{3}{\delta - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}\right\} = 0.\end{aligned} \tag{5.20}$$

Similarly, the finite quantum energy-momentum tensor reads

$$\begin{aligned}\langle\underline{\hat{T}^Q_{00}}\rangle = \frac{H^4}{32\pi^2}&\left\{6\frac{\delta - 6\xi}{\delta - 3\epsilon + \delta_H\epsilon + 3\epsilon^2} + \Big(\delta^2 - 2\delta(6\xi+1) + 24\xi + \delta(4-12\xi)\epsilon\right.\\ &\left.- 6(1-2\xi)\epsilon\Big)\log\left(\frac{H}{\mu}\right)\right\}.\end{aligned} \tag{5.21}$$

For the energy-momentum tensor we have the property

$$\langle\underline{\hat{T}^Q_{ii}}\rangle/a^2 = -\langle\underline{\hat{T}^Q_{00}}\rangle, \tag{5.22}$$

so, just like in the classical case (2.25) within our approximation we can write from the Einstein equations (5.13) the dynamical relation

$$2\epsilon H^2 = \frac{1}{M_{\rm pl}^2}\left[\frac{T_{ii}^C}{a^2} + T_{00}^C\right] = \frac{\dot{\varphi}^2}{M_{\rm pl}^2}. \tag{5.23}$$

The reason for this can be understood simply from classical physics. Like a classical potential is not present in the dynamical equation for ϵ (5.23) where one only sees the kinetic contribution, roughly the same argument applies for a quantum potential. Hence it would have been the quantum kinetic term that would have been present in the equation for ϵ, but like classically, the kinetic term is higher order compared to the potential and thus beyond our leading term approximation.

The Einstein equation involving the energy-density

$$H^2 = \frac{1}{M_{\rm pl}^2}\left[T_{00}^C + \underline{\langle \hat{T}_{00}^Q \rangle}\right] + \Lambda, \tag{5.24}$$

can be matched with the classical result at one point by renormalizing the cosmological constant to exactly cancel the quantum correction. This point can be used as an initial value for (5.23), so one may argue that most of the quantum corrections enter from the field equation of motion (5.20). However, since we can only match the energy-momentum to be the classical one at some point, in principle the first Friedmann equation will always explicitly include quantum corrected dynamics.

The $(\delta - 3\epsilon + \mathcal{O}(\epsilon^2))^{-1}$-type structure of the infrared contributions was already noticed in [138]. For some parameter values there exists a risk of obtaining a large quantum contribution, potentially making the use of the perturbative expansion ill-defined. We can derive a bound for the validity of the perturbative expansion from (5.20) by requiring the tree-level result to be much larger than the infrared term. Roughly, this gives the condition

$$\delta - 3\epsilon \gg \frac{\sqrt{\lambda}}{4\pi}. \tag{5.25}$$

Infrared divergences are frequently encountered in finite temperature field theory and usually imply that one must improve the perturbative approximation. We will achieve this by effectively re-summing the series, which can be done by instead of a free propagator using one that includes certain amounts quantum effects. What we would like to obtain is an expression that is regular at the limit $\epsilon, \delta \to 0$. For our calculation re-summing the diagrams means that instead of using an effective mass with only terms from the classical potential as defined in (2.32), we also include loop corrections in it. For this purpose we will use a systematic approach where this is achieved by using an effective action formed with two-particle-irreducible (2PI) Feynman diagrams.

5.4 One-loop 2PI approximation

The 2PI effective action approach is a systematic way of summing to infinite order a finite number of distinct topological classes of diagrams, shown to be renormalizable in [157]. This is achieved by writing and solving a self-consistent equation for the propagator. In practice this means that one writes an equation for the propagator and its effective mass in such a way that a pertubative expansion, like the one used in deriving the one-loop propagator in (3.6), is not used in any step of the calculation once the approximation

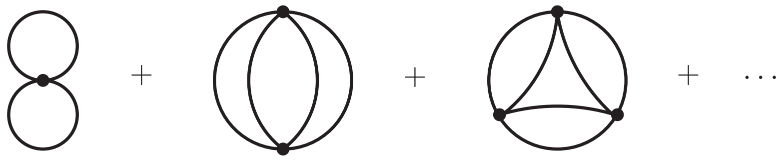

Figure 5.1: Graphs to be included in $\Gamma_2[\varphi, G, g^{\mu\nu}]$ up to four-loop order for the case of zero mean field φ.

scheme is set, i.e. the topologically distinct classes of diagrams to be included are chosen. In order to find the self-consistent equation for the propagator, one must write an effective action, with the propagator being a dynamical variable, like the field expectation value φ. It can be shown that the 2PI effective action will only include diagrams that are 2-particle-irreducible, hence the name. A recent review of the technique may be found in [10].

The 2PI effective action can be derived in a similar fashion as the effective action we used in section 3 for the Schwinger-DeWitt expansion. As mentioned, in the 2PI approach the propagator $G(x, y)$ is a variable of the action having its own self-consistent equation of motion. In direct analogy with the standard effective action approach in (3.1), one may derive the 2PI effective action by introducing a source term for the propagator, $R(x, y)$ and performing Legendre transformation with respect to the *two* sources $J(x)$ and $R(x, y)$. The result can be conveniently parametrized as [10]

$$\Gamma_{2\mathrm{PI}}[\varphi, G, g^{\mu\nu}] = S_g[g^{\mu\nu}] + S_m[\varphi, g^{\mu\nu}] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\left[G_0^{-1}G\right] + \Gamma_2[\varphi, G, g^{\mu\nu}], \quad (5.26)$$

where we now have a dependence on the yet undetermined full propagator G and the free propagator G_0, which is the one-loop approximation from (3.6) and for the φ^4 theory defined in (4.8) and can be written as

$$iG_0^{-1}(x, y) = \frac{\delta S_m[\varphi, g^{\mu\nu}]}{\delta\varphi(x)\delta\varphi(y)} = -\sqrt{-g}\left(-\Box_y + m_0^2 + \xi_0 R + \frac{\lambda_0}{2}\varphi^2\right)\delta(x-y). \quad (5.27)$$

We now have three equations of motion all derivable via variation

$$\frac{\delta\Gamma_{2\mathrm{PI}}[\varphi, G, g^{\mu\nu}]}{\delta\varphi(x)} = 0, \qquad \frac{\delta\Gamma_{2\mathrm{PI}}[\varphi, G, g^{\mu\nu}]}{\delta g^{\mu\nu}(x)} = 0, \qquad \frac{\delta\Gamma_{2\mathrm{PI}}[\varphi, G, g^{\mu\nu}]}{\delta G(x, y)} = 0. \quad (5.28)$$

Note that had we set $\Gamma_2 = 0$, the propagator equation of motion in (5.28) would have given the solution $G = G_0$ and the 2PI effective action would have coincided with the one-loop approximation in (3.2).

The quantity $\Gamma_2[\varphi, G, g^{\mu\nu}]$ depends on the approximation used and contains the essential non-perturbatice characteristics of the method. We will use a truncation at the first non-trivial order in the 2PI expansion, which is generally referred to as the Hartree approximation. At the level of the action, it amounts to including the only 2-loop "figure-8" vacuum diagram, which is the first diagram in Fig. 5.1. This is the simplest 2-particle-irreducible approximation, but still gives the right kind of re-summation behavior that is needed for taming the infrared enhancement. Hence we write

$$\Gamma_2[\varphi, G, g^{\mu\nu}] = -\frac{\lambda}{8}\int d^n x\sqrt{-g}\, G(x, x)^2. \quad (5.29)$$

For our φ^4 theory, the 2PI action from (5.26) is

$$\begin{aligned}\Gamma_{\rm 2PI}[\varphi, G, g^{\mu\nu}] &= S_g[g^{\mu\nu}] - \frac{1}{2}\int d^n x\sqrt{-g}\,\left[\partial_\mu\varphi\partial^\mu\varphi + m_0^2\varphi^2 + \xi_0 R\varphi^2 + 2\frac{\lambda_0}{4!}\varphi^4\right] \\ &\quad - \frac{i}{2}\mathrm{Tr}\ln G - \frac{1}{2}\int d^n x\sqrt{-g}\,\left(\nabla_{x,\mu}\nabla^\mu_y + m_1^2 + \xi_1 R + \frac{\lambda_1}{2}\varphi^2\right) G(x,y)\Big|_{x\to y} \\ &\quad - \frac{\lambda_2}{8}\int d^n x\sqrt{-g}\, G(x,x)^2 \\ &\equiv S_g[g^{\mu\nu}] + \Gamma_{{\rm 2PI},m}[\varphi, G, g^{\mu\nu}] \end{aligned} \tag{5.30}$$

We have explicitly written different bare couplings for each contribution in the 2PI action, because in general some of these couplings have differing counter term contributions [10].

5.4.1 2PI equation of motion for the field

Next we will solve the propagator equation of motion, whose solution is needed for the field equation of motion. The equations from (5.28) are

$$\left[-\Box + m_0^2 + \xi_0 R + \frac{\lambda_0}{6}\varphi^2 + \frac{\lambda_1}{2}G(x,x)\right]\varphi = 0 \tag{5.31}$$

$$\left[-\Box_x + m_1^2 + \xi_1 R + \frac{\lambda_1}{2}\varphi^2 + \frac{\lambda_2}{2}G(x,x)\right]G(x,y) = -i\frac{\delta(x-y)}{\sqrt{-g}}. \tag{5.32}$$

Consistent renormalization for the bare parameters in equations (5.31 - 5.32) gives a relation for the divergent counter terms

$$\delta m_0^2 = \delta m_1^2, \qquad \delta\xi_0 = \delta\xi_1, \qquad \delta\lambda_1 = \delta\lambda_2, \qquad \delta\lambda_0 = 3\delta\lambda_1 \tag{5.33}$$

and we also make the choices

$$m_0^2 = m_1^2, \qquad \xi_0 = \xi_1, \qquad \lambda_1 = \lambda_2, \qquad \lambda_0 = \lambda + \delta\lambda_0, \qquad \lambda_2 = \lambda + \delta\lambda_2, \tag{5.34}$$

so that all the above counter terms have the property $c_i = c + \delta c_i$. The crucial quantity in this approximation is again the effective mass, which in contrast to (2.32) is now defined by equation (5.32) as

$$M_{\rm 2PI}^2 \equiv m_1^2 + \xi_1 R + \frac{\lambda_1}{2}\varphi^2 + \frac{\lambda_2}{2}G(x,x), \tag{5.35}$$

If, as in section 5.1, we assume that $M_{\rm 2PI}$ is approximately constant it is easy to show we can use the mode defined in section (5.1) for equation (5.32) with the replacement $M \to M_{\rm 2PI}$. However, before we can write and solve the propagator equation, we must remove all the divergences coming from the loop $G(x,x)$.

Deriving the 2PI counter terms in the Hartree approximation is a standard calculation in Minkowski space and the generalization to dynamical space is straightforward. It is convenient first to use the result from (5.18) to write the loop contribution as

$$G(x,x) = \frac{-M_{\rm 2PI}^2 + \frac{R}{6}}{8\pi^2(4-n)} + \mathcal{F}, \tag{5.36}$$

where $\mathcal{F}$ is a finite contribution. We can then write the algebraic equation for the self-consistent mass in (5.35) as

$$\begin{aligned} M_{2\mathrm{PI}}^2 &= m^2 + \xi R + \frac{\lambda}{2}\varphi^2 + \lambda\frac{\mathcal{F}}{2} \\ &+ \left\{ \delta m_0^2 + \delta\xi_0 R + \frac{\delta\lambda_2}{2}\varphi^2 - (\delta\lambda_2 + \lambda)\frac{M_{2\mathrm{PI}}^2 - R/6}{16\pi^2(4-n)} + \delta\lambda_2\frac{\mathcal{F}}{2} \right\}, \end{aligned} \tag{5.37}$$

If we impose the condition that the expression in the curly brackets in (5.37) vanishes we can write the above as

$$\begin{aligned} &\left[\delta m_0^2 - (\lambda + \delta\lambda_2)\frac{m^2}{16\pi^2(4-n)}\right] + R\left[\delta\xi_0 - (\lambda + \delta\lambda_2)\frac{(\xi - 1/6)}{16\pi^2(4-n)}\right] \\ &- \left(\frac{\varphi^2}{2} + \frac{\mathcal{F}}{2}\right)\left[(\lambda + \delta\lambda_2)\frac{\lambda}{16\pi^2(4-n)} - \delta\lambda_2\right] = 0. \end{aligned} \tag{5.38}$$

Setting all the angular brackets separately to zero this gives us a set of counter terms, which can be used in (5.32) to remove all the divergences coming from $G(x,x)$. Then we can set $n = 4$ in (5.35) and derive the equation for the effective mass, also known as the gap equation

$$M_{2\mathrm{PI}}^2 = \tilde{M}^2 + \left(\frac{3\tilde{\lambda}}{16\pi^2}\right)\frac{H^2}{M_{2\mathrm{PI}}^2/H^2 - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}, \tag{5.39}$$

where we have defined $\tilde{M}^2 = \tilde{m}^2 + \tilde{\xi}R + \frac{\tilde{\lambda}}{2}\varphi^2$ and the running constants[3]

$$\tilde{\lambda} = \frac{\lambda}{1 - \frac{\lambda}{16\pi^2}\log\left(\frac{H}{\mu'}\right)}, \quad \tilde{m}^2 = \frac{m^2}{1 - \frac{\lambda}{16\pi^2}\log\left(\frac{H}{\mu'}\right)}, \quad \tilde{\xi} = \frac{1}{6} - \frac{1/6 - \xi}{1 - \frac{\lambda}{16\pi^2}\log\left(\frac{H}{\mu'}\right)}. \tag{5.40}$$

It is noteworthy that this running behavior is similar to the running constants obtained when using renormalization group improved effective action in the 1PI approximation [47]. In a sense, the 2PI approximation automatically includes RG improvement and running couplings. The solution for the effective mass is

$$M_{2\mathrm{PI}}^2 = H^2\left\{\frac{\tilde{\delta} + 3\epsilon - 3\epsilon^2 - \delta_H\epsilon}{2} + \sqrt{\left(\frac{\tilde{\delta} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}{2}\right)^2 + \frac{3\tilde{\lambda}}{16\pi^2}}\right\} + \mathcal{O}(4-n), \tag{5.41}$$

where we used the definition $\tilde{\delta} \equiv \tilde{M}^2/H^2$. Having solved for the effective mass, the renormalized field equation of motion (5.31) now reads

$$\left[-\Box - \frac{\lambda}{3}\varphi^2 + M_{2\mathrm{PI}}^2\right]\varphi = 0. \tag{5.42}$$

If we take the limits where the perturbative expansion is valid, i.e (5.25) we can write the effective mass as

$$M_{2\mathrm{PI}}^2 \approx M^2 + \frac{\lambda H^2}{16\pi^2}\left\{(\delta + \epsilon - 2)\log\left(\frac{H}{\mu'}\right) + \frac{3}{\delta - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon}\right\} \tag{5.43}$$

and hence equation (5.42) coincides with the one-loop field equation (5.20) in this limit.

[3] Now we must use the scale μ' defined as $\mu' = \mu\exp\left\{\frac{1}{4}\left[1 - 2\gamma_e + 2\log(\pi)\right]\right\}$, since this is what the exact calculation gives, as shown in **III**. We could previously set μ' to μ, since as mentioned in section 5.3 the additional terms vanish upon renormalization. Now our expansion is in terms of $H^2\delta_{2\mathrm{PI}}$, so our previous argument fails.

5.4.2 2PI Einstein equation

By variation we get from (5.30) the energy-momentum tensor

$$\begin{aligned} & T_{\mu\nu}^{\mathrm{2PI}} = -\frac{2}{\sqrt{-g}}\frac{\delta\Gamma_{\mathrm{2PI},m}[\varphi,G,g^{\mu\nu}]}{\delta g^{\mu\nu}} \\ \Leftrightarrow\quad & T_{\mu\nu}^{\mathrm{2PI}} = \partial_\mu\varphi\partial_\nu\varphi - \frac{g_{\mu\nu}}{2}\Big(\partial_\rho\varphi\partial^\rho\varphi + m_0^2\varphi^2 + 2\frac{\lambda_0}{4!}\varphi^4\Big) + \langle\hat{T}_{\mu\nu}^{Q}\rangle_* \\ & \qquad + g_{\mu\nu}\frac{\lambda_2}{8}G^2(x,x) + \frac{g_{\mu\nu}}{2}\xi_0 RG(x,x) + \xi_0\Big(G_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big)(\varphi^2 + G(x,x)), \end{aligned} \tag{5.44}$$

where $\langle\hat{T}_{\mu\nu}^{Q}\rangle_*$ denotes the one-loop energy-momentum tensor defined in (5.19) with M replaced by M_{2PI} defined in (5.41) and without the explicitly ξ-dependent piece. In order to find an explicit result for the energy-momentum, we can use (5.35) to express the $G(x,x)$ contributions and the 2PI counter terms derived in section 5.4.1 along with the one-loop expression (5.18). After some algebra this gives

$$\begin{aligned} T_{\mu\nu}^{\mathrm{2PI}} &= -g_{\mu\nu}\left[\frac{1}{2}\partial_\rho\varphi\partial^\rho\varphi - \frac{\lambda}{12}\varphi^4 - \frac{M_{\mathrm{2PI}}^2\left(M_{\mathrm{2PI}}^2 - 2M^2\right)}{2\lambda}\right] \\ &+ \partial_\mu\varphi\partial_\nu\varphi + 2\frac{\xi}{\lambda}\Big[R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big]M_{\mathrm{2PI}}^2 \\ &- g_{\mu\nu}\frac{H^4}{32\pi^2}\left[\frac{6\delta_{\mathrm{2PI}}}{\delta_{\mathrm{2PI}} - 3\epsilon} + \left(\delta_{\mathrm{2PI}}^2 - 2\delta_{\mathrm{2PI}} - 6\epsilon\right)\log\left(\frac{H}{\mu}\right)\right], \qquad \delta_{\mathrm{2PI}} \equiv \frac{M_{\mathrm{2PI}}^2}{H^2}, \end{aligned} \tag{5.45}$$

where we have neglected all terms that are multiples of the gravitational counter terms in (2.52) since they only give constant shifts in the renormalization counter terms and are thus physically irrelevant. Covariant conservation of (5.45) is consistent with the 2PI field equation of motion (5.42) within our approximation, which may be shown by applying with ∇_μ on $(T^{\mu\nu})^{\mathrm{2PI}}$. Taking the 1PI limit by writing

$$M_{\mathrm{2PI}}^2 \approx M^2 + \frac{\lambda}{2}\langle\hat{\phi}^2\rangle, \tag{5.46}$$

as in (5.43) and expanding (5.45) to 1-loop order we find agreement with the 1PI 1-loop results in section 5.3. The surprising thing is that there is no need for any gravitational counter terms for removing the divergences, as the 2PI counter terms for $\delta\lambda$, δm^2 and $\delta\xi$ are enough to render the energy-momentum finite. Of course we might still need additional gravitational counter terms in order to have the appropriate *finite* parts of the 2PI energy-momentum.

We can simplify the above expression by using the gap equation (5.39). Again ignoring terms that vanish after renormalization, this gives

$$T_{\mu\nu}^{\mathrm{2PI}} = -\frac{g_{\mu\nu}}{2}\partial_\rho\varphi\partial^\rho\varphi + \partial_\mu\varphi\partial_\nu\varphi + \frac{2\xi}{\lambda}\Big[R_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\Big]M_{\mathrm{2PI}}^2 - g_{\mu\nu}W_{\mathrm{2PI}}(\varphi,H,\epsilon), \tag{5.47}$$

where we have defined the potential

$$W_{\mathrm{2PI}}(\varphi,H,\epsilon) \equiv -\frac{\lambda}{12}\varphi^4 + \frac{M_{\mathrm{2PI}}^4}{2\tilde{\lambda}} + \left(\frac{1}{\lambda} - \frac{1}{\tilde{\lambda}}\right)M_{\mathrm{2PI}}^2 H^2 + \frac{3\epsilon H^4}{\tilde{\lambda}}. \tag{5.48}$$

We note that this potential in general differs from the effective potential that is the non-kinetic part of the field equation of motion. This is because quantum corrections necessarily introduce terms with metric dependence, which makes the standard formalism defined via simply one $V(\varphi)$ inapplicable. By using (5.47 – 5.48) we can again derive the 2PI Friedmann equations from the Einstein equation (5.13)

$$3H^2 = \frac{1}{M_{\rm pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + 6\frac{\xi}{\lambda}(H\partial_t - H^2)M_{\rm 2PI}^2 + W_{\rm 2PI}(\varphi, H, \epsilon)\right] \tag{5.49}$$

$$a^2(-3H^2 + 2\epsilon H^2) = \frac{a^2}{M_{\rm pl}^2}\left[\frac{1}{2}\dot{\varphi}^2 + 6\frac{\xi}{\lambda}\bigg(-\frac{1}{3}(2H\partial_t + \partial_t^2) + H^2\bigg)M_{\rm 2PI}^2 - W_{\rm 2PI}(\varphi, H, \epsilon)\right]. \tag{5.50}$$

These expressions also give the classical dynamical (5.23) relation at $\xi = 0$.

5.5 Finite renormalization

Up until this point, in all the calculations of this section, as far as renormalization is concerned, we have only been interested in making the expressions finite. Next we proceed to obtain a physical interpretation of the constants. We now use the method of **II** explained in section 4 in order to fix the finite parts of the counter terms. To achieve this we may split all the constants to a physical part and a finite counter term as $c \to c_{\mathbf{ph}} + \tilde{\delta}c$. Since in the end we are interested only in the minimally coupled case, we will set $\xi_{\mathbf{ph}} = 0$

To begin, we write the field equation of motion for the field symbolically as

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V(\varphi)}{\partial \varphi} = 0, \tag{5.51}$$

where the potential $V(\varphi)$ is split into a physical part and finite counter terms:

$$V(\varphi) \equiv V_{\mathbf{ph}}(\varphi) + \tilde{\delta}V(\varphi), \tag{5.52}$$

with

$$\tilde{\delta}V(\varphi) = \tilde{\delta}\sigma\varphi + \frac{\tilde{\delta}m^2}{2}\varphi^2 + \frac{\tilde{\delta}\xi}{2}R\varphi^2 + \frac{\tilde{\delta}\eta}{3!}\varphi^3 + \frac{\tilde{\delta}\lambda}{4!}\varphi^4. \tag{5.53}$$

For consistency we have introduced counter terms for one- and three-point couplings, even though these terms are not present classically and they are not needed for removing the quantum divergences. Nevertheless, they will give non-zero contributions when renormalization is performed at non-zero scale choices for φ and $\dot{a}$.

Here, in contrast to section 4.3, the quantity of interest is the scalar field potential $V(\varphi)$ instead of the energy-density, which we match to the classical potential

$$V_C(\varphi) = \frac{1}{2}m_{\mathbf{ph}}^2\varphi^2 + \frac{\lambda_{\mathbf{ph}}}{4!}\varphi^4 \tag{5.54}$$

at the renormalization point

$$\mu_0 = (\varphi_0,\ H_0,\ \epsilon_0,\ \dot{\varphi}_0,\ \ddot{\varphi}_0), \tag{5.55}$$

expressible with the conditions

$$\begin{aligned}
\frac{\partial V(\varphi)}{\partial \varphi}\bigg|_{\mu_0} &= m_{\mathbf{ph}}^2 \varphi_0 + \frac{\lambda_{\mathbf{ph}} {\varphi_0}^3}{6}, & \frac{\partial^2 V(\varphi)}{\partial \varphi^2}\bigg|_{\mu_0} &= m_{\mathbf{ph}}^2 + \frac{\lambda_{\mathbf{ph}} {\varphi_0}^2}{2}, \\
\frac{\partial^3 V(\varphi)}{\partial \varphi^3}\bigg|_{\mu_0} &= \lambda_{\mathbf{ph}} \varphi_0, & \frac{\partial^4 V(\varphi)}{\partial \varphi^4}\bigg|_{\mu_0} &= \lambda_{\mathbf{ph}}, \\
\frac{\partial^4 V(\varphi)}{\partial H^2 \partial \varphi^2}\bigg|_{\mu_0} &= 0.
\end{aligned} \tag{5.56}$$

With this procedure we can solve for the finite parts of the counter terms to get the renormalized equation of motion for both the 1PI equation in (5.20) and the 2PI equation in (5.42). The important difference in the two approaches, in addition to using different potentials, is that in the 1PI approximations the counter terms enter only through the classical contribution, where as in 2PI every constant will contain a counter term. In the 1PI approximation, we can conveniently parametrize the result as

$$\begin{aligned}
\ddot{\varphi} + 3H\dot{\varphi} + \Delta\sigma + (m_{\mathbf{ph}}^2 + \Delta m^2)\varphi + \Delta\xi R\varphi + \frac{1}{2}\Delta\eta\varphi^2 + \frac{\lambda_{\mathbf{ph}} + \Delta\lambda}{6}\varphi^3 \\
+ \frac{\lambda_{\mathbf{ph}} \varphi H^2}{16\pi^2} \left\{ (\delta_{\mathbf{ph}} + \epsilon - 2) \log\left(\frac{H}{H_0}\right) + \frac{3}{\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H \epsilon} \right\} = 0,
\end{aligned} \tag{5.57}$$

where $\delta_{\mathbf{ph}}$ denotes δ with all the constants replaced by the physical ones: $m^2 \to m_{\mathbf{ph}}^2$, etc. The quantum induced Δ-terms are finite constants depending on the physical parameters $m_{\mathbf{ph}}^2$ and $\lambda_{\mathbf{ph}}$ and the renormalization point μ_0. In general, expressions for the Δ's are rather complicated, as one may see from the appendixes of **III**.

Next, we consider the Friedman equations, which upon including finite counter terms read

$$3H^2 = \frac{1}{M_{\rm pl}^2} \left[T_{00} + \tilde{\delta} T_{00} \right], \tag{5.58}$$

$$a^2\big(-3H^2 + 2\epsilon H^2 \big) = \frac{1}{M_{\rm pl}^2} \left[T_{ii} + \tilde{\delta} T_{ii} \right]. \tag{5.59}$$

We choose to renormalize the cosmological constant such that at the renormalization point $\varphi = \varphi_0$ the energy density coincides with the classical result:

$$T_{00}\big|_{\mu_0} = T_{00}^C\big|_{\mu_0} = \frac{1}{2}\dot{\varphi}_0^2 + V_C(\varphi_0). \tag{5.60}$$

If we further choose the natural condition that the counter term for $G_{\mu\nu}$ vanishes, we can write the 1PI Friedmann equations as

$$3H^2 = \frac{1}{M_{\rm pl}^2} \left[T_{00}^C + \Delta V_C(\varphi) + \frac{3H^2 \delta_{\mathbf{ph}}}{16\pi^2(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H \epsilon)} - T_{00}^Q\Big|_{\mu_0} \right], \tag{5.61}$$

$$\epsilon H^2 = \frac{\dot{\varphi}^2}{2M_{\rm pl}^2}, \tag{5.62}$$

where we have defined

$$\Delta V_C(\varphi) = \Delta\sigma\varphi + \frac{\Delta m^2}{2}\varphi^2 + \frac{\Delta\xi}{2}R\varphi^2 + \frac{\Delta\eta}{3!}\varphi^3 + \frac{\Delta\lambda}{4!}\varphi^4 \tag{5.63}$$

and for simplicity have left out the (sub-leading) logarithmic terms.

Implementing equations (5.56) is in principle perfectly possible in 2PI, but due to the highly non-linear structure of the resulting equations this may require the use of numerical methods. Fortunately for the standard chaotic inflation models we are interested in, we can easily show that the 1PI approximation gives sensible results.

5.6 Size of the quantum corrections

As discussed in section 2.2 for chaotic inflation, which is under study here, the physical constants must be extremely small in order to have a sufficiently flat potential. We can give a rough estimate for the constants for theories with a quadratic or quartic potential by using the amplitude of the spectrum (2.12) given in terms of slow-roll parameters [38] including only the leading $1/\epsilon$ contribution

$$\mathcal{P}(k) \approx \frac{V(\varphi)}{24\pi^2 M_{\rm pl}^4}\epsilon_V^{-1} \tag{5.64}$$

and current Planck data [4] at 60 e-folds before the end of inflation. For a massless $(\lambda/4!)\varphi^4$ potential by using the formulae of section (2.3) we get roughly $\lambda_{\mathbf{ph}} \sim 10^{-12}$, from which we can deduce with the help of (5.25) that the 1PI results are perfectly adequate.

For our renormalization scales, we first approximate that at the renormalization point we can use the terminal velocity condition $\ddot{\varphi} = 0$ in order to express all the scales in (5.55) as functions of just one scale φ_0 and then make the choice $\varphi_0 = 22M_{\rm pl}$, such that φ_0 corresponds to approximately 60 e-foldings before the end of inflation. For the field equation of motion (5.57) we can study the magnitude of the quantum correction by comparing the quantum induced terms to the tree-level ones. With these choices, for example we have $\Delta\lambda/\lambda_{\mathbf{ph}} \sim 10^3\lambda_{\mathbf{ph}}$, which is negligibly small. The other Δ's and the quantum terms in the second line of the field equation (5.57) give similar size corrections, so we can conclude that the quantum corrections may be ignored to a good approximation. Similarly, for chaotic inflation with a potential $(m^2/2)\varphi^2$ we trivially obtain the classical field equation of motion, since all quantum corrections are proportional to the interaction constant $\lambda_{\mathbf{ph}}$.

It would thus seem that for the standard models of chaotic inflation the quantum corrections are by and large unobservable, at least for the field equation of motion. To be sure that a similar result is valid also for the Friedmann equations, we will calculate the quantum correction for the slow-roll parameter ϵ. Using the slow-roll formulae of section 2.3, we can write ϵ as

$$\epsilon = \frac{(\partial_\varphi V)^2}{18M_{\rm pl}^2 H^4}. \tag{5.65}$$

An easy way of getting a first approximation for the size of the quantum corrections is to split the effective potential in (5.51) into classical and quantum parts: $V = V_C + V_Q$, with a similar split for the energy density given by the right hand side of (5.58), and then using the tree-level results *inside* the quantum contributions. This allows us to express the slow-roll ϵ as a classical and a quantum correction from (5.65)

$$\epsilon = \epsilon_C + \epsilon_Q \tag{5.66}$$

where the leading quantum correction is given by

$$\epsilon_Q = \left[\frac{2}{M_{\mathbf{ph}}^2\varphi - \frac{\lambda_{\mathbf{ph}}}{3}\varphi^3}\left(\Delta V_C'(\varphi) + \frac{3\lambda_{\mathbf{ph}}\varphi H_C^2}{16\pi^2(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon)_C)}\right)\right.$$
$$\left. - \frac{2}{V_C}\left(\Delta\Lambda + \Delta V_C(\varphi) + \frac{3H^4\delta_{\mathbf{ph}}}{16\pi^2(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon)_C)}\right)\right]\epsilon_C \tag{5.67}$$

with the definitions $\Delta\Lambda \equiv -T_{00}^{Q}\big|_{\mu_0}$ and $\Delta V_C(\varphi)$ from (5.63). As for the field equation of motion, we will evaluate the size of the quantum corrections in two opposite limits for the potential with either only a mass term $(m^2/2)\varphi^2$ or a quartic self-interaction term $(\lambda/4!)\varphi^4$. Again from (5.64) and [4] we get the estimate $m_{\mathbf{ph}}^2 \sim 10^{-11}M_{\mathrm{pl}}^2$ for the massive non-interacting theory. Furthermore, in this limit we find that

$$\delta_C - 3\epsilon_C \sim \mathcal{O}(\epsilon_C^2). \tag{5.68}$$

The leading correction comes from the last term of (5.67) and with the help of the tree-level slow-roll parameters, it can be written as

$$\frac{3}{16\pi^2(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon)_C} \cdot \frac{H_C^4\delta_{\mathbf{ph}}}{V_C} \approx \frac{1}{16\pi^2}\frac{m_{\mathbf{ph}}^2}{M_{\mathrm{pl}}^2}\frac{(2N_C+1)^2}{3}, \tag{5.69}$$

where N_C is the (classical) number of e-folds from (2.26). Again this is totally negligible for the physically interesting scales $N \lesssim 100$. For the massless self-coupled case we have $\lambda_{\mathbf{ph}} \sim 10^{-12}$ which give for the IR enhancement factor

$$\frac{1}{(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon)_C} \approx \frac{2}{3\epsilon_C}. \tag{5.70}$$

Again tree-level slow-roll considerations give us the estimate for the largest terms of (5.67), which can be approximated as

$$\frac{3\lambda_{\mathbf{ph}}}{16\pi^2(\delta_{\mathbf{ph}} - 3\epsilon + 3\epsilon^2 + \delta_H\epsilon)_C} \cdot \frac{H_C^2}{M_{\mathbf{ph}}^2} \approx \frac{\lambda_{\mathbf{ph}}}{16\pi^2}\frac{4(N_C+1)^2}{9}. \tag{5.71}$$

When this procedure is implemented for δ_H, one obtains a similar size estimate for the quantum correction. Hence, unless we study effects deep within inflation[4], $N_C \sim 10^6$, all the quantum corrections are negligible for the standard models of chaotic inflation.

5.7 Discussion

One of our main conclusions is that for the standard quadratic and quartic models of chaotic inflation curved space quantum corrections make little difference in practice. In hindsight, this was to be expected. After all, quantum corrections usually include additional powers of the tree-level coupling constants and for chaotic inflation they are very small. This should not overshadow the theoretical significance of **III**. We were able to generalize the previous works in [135, 136, 137, 138] for a non-static spacetime with 2PI

[4]In that region the 1PI approximation cannot be trusted due to the smallness of the slow-roll parameters and hence the results in this section are not applicable.

re-summation and provide information about the non-trivial aspects of the infrared behavior not seen by the heat kernel expansion of chapter 3. Additionally, the calculation gives a blue-print that can likely be extended to more complicated models and it is not trivially obvious that quantum corrections can always be neglected. In particular, already in **III** we saw hints that for the curvaton model the 1PI approximation leads to a divergent loop contribution, thus potentially signifying a non-trivial quantum correction and a need for improving the perturbative expansion, possibly via re-summation techniques. At the moment this is only a preliminary observation and naturally requires a detailed study before a conclusion may be reached.

Much work still lies ahead. On the phenomenological side due to the large number of various inflationary models, there are many ways of generalizing our results to more complicated models or including quantum fluctuations of the metric. These and other matters are discussed in the final chapter of this thesis.

Chapter 6

Conclusions and outlook

In this thesis we have studied the effects of quantum corrections for simple scalar field inflationary models within the framework of curved space field theory. We have approached the problem via the effective action formalism, and also at the equation of motion level, for which we devised an approach for consistent renormalization. What we were able to show was that such calculations, including the implementation of the 2PI re-summation technique in a non-static background, are perfectly feasible to perform in practice, although significantly more laborious than in the flat space context. In terms of actual models, our main focus was on chaotic inflation driven by a single scalar field with a renormalizable potential of the form (2.48). For such models, we concluded that quantum corrections are by and large unobservable. Despite of this, the curved space quantum corrections have theoretically a very interesting structure, which is not present when using field theory in Minkowski space. Hence, the natural next step would be to implement these techniques for more complicated models than the standard single field chaotic inflation, starting from the curvaton scenario. The fact that quantum corrections may be significant for curvaton models has already been observed via the stochastic method in [158].

In general, inflationary models with multiple scalar fields provide a much wider range of possibilities than single field models [159] validating their study in the hopes of experimental verification by future measurements. Generalizing our approach to models with more than one scalar field is straightforward, as should be the study of models where inflation is driven by spinors [160] or vector fields [161]. This assertion stems from realizing that the quantization of vector and spinor fields in curved spacetime has been understood for quite some time already [7, 8]. Furthermore, by having a model with more – not necessarily bosonic – fields is required by the universe to properly re-heat after inflation, making such considerations a natural generalization. Another important class of models that could be studied by the means presented here is where gravity includes higher order tensors, in particular the so-called $f(R)$ models [162]. In $f(R)$ models the gravitational action contains an arbitrary function of R. In fact, as we saw in section 2.5 in curved space field theory the higher order tensors are required for the theory to be consistent.

Potentially an even more important generalization of our results would be to include also fluctuations of the metric. It is standard knowledge that including the gravity fluctuations in the calculation of the primordial spectrum gives corrections at first order in slow-roll [38] and hence in principle these effects should be included if one wishes to obtain an accuracy at leading slow-roll order. Of course, one then needs to address the issue of nonrenormalizabilty of Einsteinian gravity. This matter is even more involved if one wishes to perform resummations also in this context. However, the possible reward for a successful resummation of also gravity fluctuations is significant as it may provide a new solution to the problem of infrared divergences in cosmological correlations [100], just like it solved the potential infrared enhancement for scalar fields in quasi-de Sitter space in chapter 5.

The renormalization procedure of chapter 4 may also be applied to problems outside of inflationary physics and it is our hope that it provides a fruitful new tool for other areas cosmology where quantum corrected curved space calculations are needed. A particularly interesting application would be the very difficult cosmological constant problem (and related matters such as its possible running [163]), where one of the main open questions is providing a renormalization condition with a clear physical interpretation for all of the constants of theory at a specific renormalization scale [134]. All in all, we hope that the calculations presented in this thesis, and more importantly in the articles **I** – **III**, will serve as not just an academic exercise, but a welcome new angle on problems of early universe physics and quantum fields in curved spaces in general. Whether or not this will turn out to be the case is, of course, for the future to decide.

Appendix A

Tensor formulae

In this thesis we will frequently need the following n-dimensional geometric tensors, defined via variation with respect to the metric $g^{\mu\nu}$

$$G_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\ R = -\frac{1}{2} R g_{\mu\nu} + R_{\mu\nu}, \tag{A.1}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\ R f(x) = \big[-\frac{1}{2} R g_{\mu\nu} + R_{\mu\nu} - \nabla_\mu \nabla_\nu + g_{\mu\nu} \Box \big] f(x), \tag{A.2}$$

$${}^{(1)}H_{\mu\nu} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\ R^2 = -\frac{1}{2} R^2 g_{\mu\nu} + 2 R_{\mu\nu} R - 2 \nabla_\mu \nabla_\nu R + 2 g_{\mu\nu} \Box R, \tag{A.3}$$

$$\begin{aligned} {}^{(2)}H_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\ R^{\mu\nu} R_{\mu\nu} \\ &= -\frac{1}{2} R_{\alpha\beta} R^{\alpha\beta} g_{\mu\nu} + 2 R_{\rho\nu\gamma\mu} R^{\rho\gamma} - \nabla_\nu \nabla_\mu R + \frac{1}{2} \Box R g_{\mu\nu} + \Box R_{\mu\nu}, \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} H_{\mu\nu} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g}\ R^{\mu\nu\sigma\delta} R_{\mu\nu\sigma\delta} \\ &= -\frac{g_{\mu\nu}}{2} R^{\alpha\sigma\gamma\delta} R_{\alpha\sigma\gamma\delta} + 2 R_\mu{}^{\rho\alpha\sigma} R_{\nu\rho\alpha\sigma} + 4 R_{\sigma\mu\gamma\nu} R^{\gamma\sigma} - 4 R_{\mu\gamma} R^\gamma{}_\nu + 4 \Box R_{\mu\nu} - 2 \nabla_\mu \nabla_\nu R. \end{aligned} \tag{A.5}$$

We will often use a spacetime with the line-element

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 d\mathbf{x}^2 \tag{A.6}$$

and therefore we will need explicit expressions in this spacetime for the term with covariant derivatives in (A.2) and the Ricci scalar R and tensor $R_{\mu\nu}$. They can be respectively written as

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu) = -\partial_0^2 - 3\frac{\dot{a}}{a}\partial_0 \tag{A.7}$$

$$(-\nabla_0\nabla_0 + g_{00}\Box)f(t) = (n-1)\frac{\dot{a}}{a}\dot{f}(t), \tag{A.8}$$

$$(-\nabla_i\nabla_i + g_{ii}\Box)f(t) = a^2\left[(2-n)\frac{\dot{a}}{a}\dot{f}(t) - \ddot{f}(t)\right], \tag{A.9}$$

$$R = 2(n-1)\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right) + (n-1)(n-4)\frac{\dot{a}^2}{a^2}, \tag{A.10}$$

$$G_{00} = \frac{(n-1)(n-2)}{2}\left(\frac{\dot{a}}{a}\right)^2, \tag{A.11}$$

$$G_{ii} = a^2(2-n)\left[\frac{(n-3)}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]. \tag{A.12}$$

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